Intermediate Math Circles
October 5, 2011
Ratios

What is a ratio?

A ratio is a relation of the amounts of two or more things whose units of measure are the same.

A ratio can be expressed in many ways. For example, if we have twenty parts coffee for three parts cream, we can represent this as a ratio in the following ways:

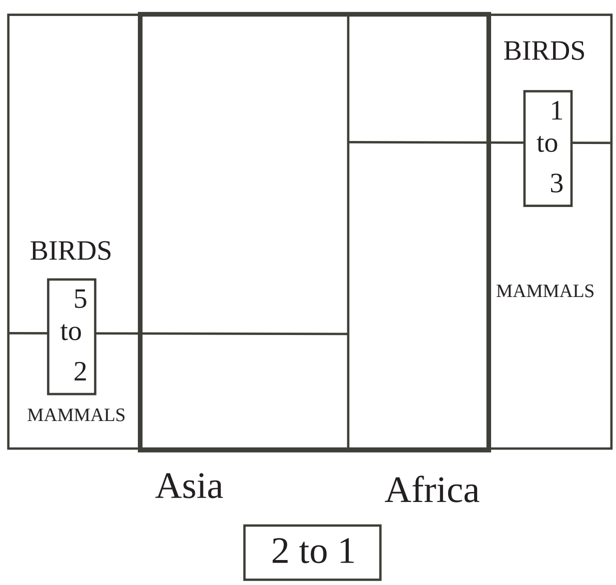
20 to 3 $\qquad$ 20 : 3 $\qquad$ $\frac{20}{3}$

As with fractions, ratios are usually written in simplified or reduced form. When in simplified form no factor, other than 1, is common to every term in the ratio and each term is an integer. The are many equivalent ratios but only one is in simplest form.

$3 : 1$, $\frac{3}{7} : \frac{1}{7}$, $1.5 : 0.5$, $6 : 2$, $12 : 4$ and $300 : 100$ are all equivalent ratios but 3:1 is the only one written in simplest form.

Problem:

A zoo enclosure is divided in the ratio of 2:1 among animals from Asia and animals from Africa. The area of the zoo that contains animals from Asia is divided in the ratio of 5:2 between birds and mammals, while the area of the zoo that contains animals from Africa is divided in the ratio of 1:3 between birds and mammals. What is the ratio of all birds in the zoo to all mammals in the zoo?



Solution:

Notice that the ratio of animals from Asia to animals from Africa to total animals at the zoo is $2:1:3$. Therefore $\frac{2}{3}$ of the animals are from Asia and $\frac{1}{3}$ of the animals are from Africa. Similarly in the Asia enclosure, the ratio of birds to mammals to total animals is $5:2:7$ so $\frac{5}{7}$ are birds and $\frac{2}{7}$ are mammals. In the Africa enclosure $\frac{1}{4}$ are birds and $\frac{3}{4}$ are mammals.

Therefore, out of the entire enclosure, the fraction of birds is

$$\left(\frac{5}{7} \times \frac{2}{3}\right) + \left(\frac{1}{4} \times \frac{1}{3}\right) = \frac{10}{21} + \frac{1}{12} = \frac{40}{84} + \frac{7}{84} = \frac{47}{84}$$

while the fraction of mammals is

$$\left(\frac{2}{7} \times \frac{2}{3}\right) + \left(\frac{3}{4} \times \frac{1}{3}\right) = \frac{4}{21} + \frac{1}{4} = \frac{16}{84} + \frac{21}{84} = \frac{37}{84}$$

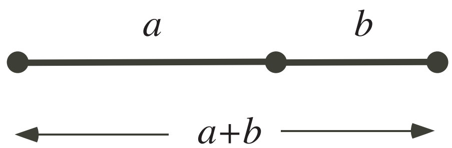
(We could have determined this ratio by subtracting $\frac{47}{84}$ from 1 giving the same result $\frac{37}{84}$.)

Therefore, in the whole zoo, the ratio of birds to mammals is $\dfrac{47}{84} : \dfrac{37}{84} = 47 : 37$.

The Golden Ratio

The discovery of the Golden Ratio is usually attributed to Pythagoras and his students in Ancient Greece. However, the first known written definition of it is by Euclid. Euclid called it the "extreme and mean ratio". He said that a line segment was cut in this ratio when the ratio of the length of the whole segment to the length of the larger part was the same as the ratio of the length of the larger part to the length of the smaller part.

Using this definition, we can calculate the numerical value of the Golden Ratio. The line segment below is split into a larger part of length a and a smaller part of length b $(a > b)$.



Let us represent the Golden Ratio by phi, ϕ.

Then we have $\phi = \dfrac{a+b}{a} = \dfrac{a}{b}$. Therefore $a = b\phi$, and we can substitute this into the equation.

$$\phi = \frac{a+b}{a} = \frac{b\phi + b}{b\phi} = \frac{b(\phi+1)}{b\phi} = \frac{\phi+1}{\phi}$$

Multiplying both sides by ϕ and moving everything to one side, we get $\phi^2 - \phi - 1 = 0$.

Using the quadratic equation $\phi = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1$, $b = -1$ and $c = -1$, we get

$$\phi = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$\phi = \frac{1 \pm \sqrt{5}}{2}$$

Since $a > b$, the golden ratio must be larger than 1, so $\phi = \dfrac{1+\sqrt{5}}{2}$.

The Golden Ratio is an example of an *irrational number*.

Using your calculator you will find that the value of the golden ratio is approximately 1.61803339887.

How is the other root related to ϕ? Numerically, $\dfrac{1-\sqrt{5}}{2} \doteq -0.6180339887$.

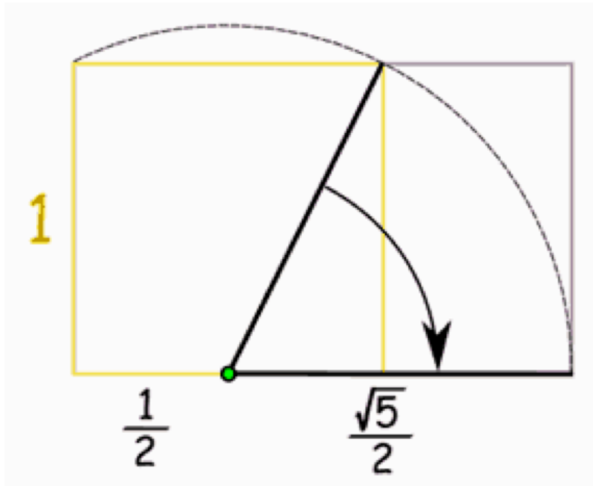
$$\frac{1-\sqrt{5}}{2} = \frac{(1-\sqrt{5})(1+\sqrt{5})}{2(1+\sqrt{5})} = \frac{1-5}{2(1+\sqrt{5})} = \frac{-4}{2(1+\sqrt{5})} = -\left(\frac{2}{1+\sqrt{5}}\right) = -\left(\frac{1}{\phi}\right)$$

The absolute value of this quantity corresponds to the length order taken in reverse order (i.e. shorter segment to longer segment). This absolute value is called Φ, capital phi. So $\Phi = \frac{1}{\phi}$.

It turns out that $\frac{1}{\phi} = \phi - 1$ and $\frac{1}{\Phi} = \Phi + 1$. These last two connections are left as exercises for the student to pursue.

If you look online for the golden ratio you will find many uses for it and many applications.

It is straight forward to construct a rectangle with sides in the golden ratio.



1. Draw a square with sides of length 1 unit.
2. Bisect the base side.
3. From the midpoint of the base draw a circle passing through the upper two vertices of the square.
4. Extend the base of the square to intersect the circle.

The length of the line segment from the bottom left vertex of the square, horizontally along the square to the point of intersection with the circle is $\frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} = \phi$ and the ratio of the sides of the rectangle is $\frac{1+\sqrt{5}}{2} : 1 = \phi : 1$.

The golden ratio has also been linked to the *Fibonacci numbers*. The first twenty-one terms of the Fibonacci sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2583, 4181, 6765, 10946$$

If you take two consecutive Fibonacci numbers, such as 5 and 8 and divide the larger by the smaller, you will get a number close to the golden ratio. As the consecutive Fibonacci numbers you choose get higher and higher, the number you get when you divide them will get closer and closer to the golden ratio. Let F_n represent the nth Fibonacci number. Then $F_{12} : F_{11} = 144 : 89 \doteq 1.617977528$ and $F_{20} : F_{19} = 6765 : 4181 \doteq 1.6180133963$. These ratios are getting extremely close to ϕ.

In general, if F_n represents the nth Fibonacci number, then this ratio of two consecutive Fibonacci numbers is represented by $\frac{F_n}{F_{n-1}}$. It has been proven that the limit of this ratio as n goes to infinity is exactly the golden ratio.

Symbolically, $\phi = \lim\limits_{n\to\infty} \dfrac{F_n}{F_{n-1}}$.