# Intermediate Math Circles
# November 09, 2011
# Problem Set - Counting III

1. How many permutations can be formed from the letters of "MNEMONIC".

   Solution
   There are 8 spots to fill with 2 M's, 2 N's, 1 E, 1 O, 1 I and 1 C.
   There are $\binom{8}{2}$ ways to place the 2 M's
   There are now $\binom{8-2}{2} = \binom{6}{2}$ ways to place the 2 N's.
   There are now 4 different letters left, to fill the remaining $6 - 2 = 4$ spots, which can be done in 4! ways.

   Using the product rule:

$$\begin{aligned}\binom{8}{2} \times \binom{6}{2} \times 4! &= \left(\frac{8!}{2!6!}\right) \times \left(\frac{6!}{2!4!}\right) \times 4! \\ &= \frac{8!}{2!2!}\end{aligned}$$

   $\therefore$ There are $\frac{8!}{2!2!} = 10\,080$ permutations that can be formed from the letters of "MNEMONIC".

2. A bit string is a sequence in which each term is either 0 or 1. For example, 0001110100 is a bit string of length 10 with six 0's and four 1's. How many bit strings of length 10 can be made with six 0's and four 1's if:

   a) there are no restrictions.

      Solution
      There are $\binom{10}{6}$ ways to play the 0's.
      There is 1 way to place the four 1's in the remaining 4 places.

      Using the product rule:

$$\begin{aligned}\binom{10}{6} \times 1 &= \left(\frac{10!}{6!4!}\right) \\ &= 210\end{aligned}$$

      $\therefore$ There are 210 bit strings of length 10 that can be made with six 0's and four 1's if there are no restrictions.

b) the bit string must begin and end with a zero.

Solution
$\underline{0}$ __ __ __ __ __ __ __ __ $\underline{0}$

There are 8 remaining spots with four 0's and four 1's to place.
There are $\binom{8}{4}$ ways to place the remaining 0's.
There is 1 way to place the four 1's in the remaining four spots.

Using the product rule:

$$\begin{aligned}\binom{8}{4} \times 1 &= \left(\frac{8!}{4!4!}\right) \\ &= 70\end{aligned}$$

$\therefore$ There are 70 bit strings of length 10 that can be made with six 0's and four 1's if the bit string must being and end with a zero.

c) the 0's must be in pairs and the 1's must be in pairs. For example, 0011000011 would be such a string. Notice that 2 pairs of pairs of zeros can be together.

Solution
Since there are six 0's, there are $\frac{6}{2} = 3$ pairs of 0's.
Similarly, since there are four 1's, there are $\frac{4}{2} = 2$ pairs of 1's
Since there were 10 spots to fill, if we are filling them with pairs there are now $\frac{10}{2} = 5$ spots to fill.

There are $\binom{5}{3}$ ways to place the 3 pairs of 0's. There is only 1 way to place the 2 pairs of 1's in the remaining two places.

Using the product rule:

$$\begin{aligned}\binom{5}{3} \times 1 &= \left(\frac{5!}{3!2!}\right) \\ &= 10\end{aligned}$$

$\therefore$ There are 10 bit strings of length 10 that can be made with six 0's and four 1's if the 0's must be in pairs and the 1's must be in pairs.

3. At this time of the year many people string lights across the front of their homes. A certain home owner had a string of lights with 24 spaces for bulbs. He had 6 red bulbs, 6 green bulbs, 6 yellow bulbs and 6 blue bulbs. How many different ways can the home owner put all of the bulbs in the sockets if:

   a) there are no restrictions?

   Solution
   There are $\binom{24}{6}$ ways to place the 6 red bulbs.
   There are now $\binom{24-6}{6} = \binom{18}{6}$ ways to place the 6 green bulbs.
   There are now $\binom{18-6}{6} = \binom{12}{6}$ ways to place the 6 yellow bulbs.
   There is now only 1 way to place the 6 blue bulbs in the remaining 6 places.

   Using the product rule:

$$\begin{aligned}\binom{24}{6}\times\binom{18}{6}\times\binom{12}{6}\times 1 &= \left(\frac{24!}{6!18!}\right)\times\left(\frac{18!}{6!12!}\right)\times\left(\frac{12!}{6!6!}\right)\\ &= \left(\frac{24!}{6!6!6!6!}\right)\end{aligned}$$

   $\therefore$ There are $\left(\frac{24!}{6!6!6!6!}\right)$ ways to put all the bulbs in the sockets if there are no restrictions.

   b) a red bulb must always be in a position to the immediate right of a green bulb as he looked from left to right?

   Solution
   There are 6 pairs in the form GR
   Since the red and green lights are in pairs there are now 24-6=18 places to fill.

   There are $\binom{18}{6}$ ways to place these pairs.
   There are now $\binom{18-6}{6} = \binom{12}{6}$ ways to place the 6 yellow bulbs.
   There is now only 1 way to place the 6 blue bulbs in the remaining 6 places.

   Using the product rule:

$$\begin{aligned}\binom{18}{6}\times\binom{12}{6}\times 1 &= \left(\frac{18!}{6!12!}\right)\times\left(\frac{12!}{6!6!}\right)\\ &= \left(\frac{18!}{6!6!6!}\right)\\ &= 17\ 153\ 136\end{aligned}$$

   $\therefore$ There are $\left(\frac{18!}{6!6!6!}\right) = 17\ 153\ 136$ ways to put all the bulbs in the sockets if there must be a red bulb to the immediate right of every green bulb.

c) all lights of the same colour must be together?

Solution
By making groups of the same colour, there are 4 groups of 4 different colours.
There are 4! ways to arrange these groups.
$\therefore$ There are $4! = 24$ ways to put all the bulbs in the sockets if all the lights of the same colour must be together.

4. Using all of the letters of the name *Mississippi*, how many permutations can be formed if:

a) there are no restrictions?

Solution
There are 11 spots to place 1 M, 4 I's, 4 S's, and 2 P's.
There are $\binom{11}{1}$ ways to place the M.
There are now $\binom{11-1}{4} = \binom{10}{4}$ ways to place the 4 I's.
There are now $\binom{10-4}{4} = \binom{6}{4}$ ways to place the 4 S's.
There is now 1 way to place the 2 P's in the remaining 2 spots.

Using the product rule:

$$\begin{aligned}\binom{11}{1} \times \binom{10}{4} \times \binom{6}{4} \times 1 &= \left(\frac{11!}{1!10!}\right) \times \left(\frac{10!}{4!6!}\right) \times \left(\frac{6!}{4!2!}\right)\\ &= \left(\frac{11!}{4!4!2!}\right)\\ &= 34\ 650\end{aligned}$$

$\therefore$ There are $\left(\frac{11!}{4!4!2!}\right) = 34\ 650$ permutations that can be formed if there are no restrictions.

b) the permutation must begin with $M$?

Solution
M __ __ __ __ __ __ __ __ __ __

There are now 10 spots to fill with 4 I's, 4 S's and 2 P's.
There are $\binom{10}{4}$ ways to place the 4 I's.
There are now $\binom{10-4}{4} = \binom{6}{4}$ ways to place the 4 S's.
There is now 1 way to place the 2 P's in the remaining 2 spots.

Using the product Rule:

$$\begin{aligned}\binom{10}{4} \times \binom{6}{4} \times 1 &= \left(\frac{10!}{4!6!}\right) \times \left(\frac{6!}{4!2!}\right)\\ &= \left(\frac{10!}{4!4!2!}\right)\\ &= 3\ 150\end{aligned}$$

$\therefore$ There are $\left(\frac{10!}{4!4!2!}\right) = 3\ 150$ permutations that can be formed if the permutation must begin with M.

c) the two $p$'s must be together?

Solution
Pairing up the 2 P's as 1, we are left with 10 spots to fill with 1 M, 4 I's, 4 S's, and 1 pair of P's.
There are $\binom{10}{1}$ ways to place the M.
There are now $\binom{10-1}{4} = \binom{9}{4}$ ways to place the 4 I's.
There are now $\binom{9-4}{4} = \binom{5}{4}$ ways to place the 4 S's.
There is now 1 way to place the pair of P's in the remaining 1 spot.

Using the product rule:

$$\begin{aligned}\binom{10}{1} \times \binom{9}{4} \times \binom{5}{4} \times 1 &= \left(\frac{10!}{1!9!}\right) \times \left(\frac{9!}{4!5!}\right) \times \left(\frac{5!}{4!1!}\right) \\ &= \left(\frac{10!}{4!4!}\right) \\ &= 6\ 300\end{aligned}$$

$\therefore$ There are $\left(\frac{10!}{4!4!}\right) = 6\ 300$ permutations that can be formed if the two P's must be together.

d) the two $p$'s must never be together?

Solution
From part a), it can be seen that there are $\left(\frac{11!}{4!4!2!}\right) = 34\ 650$ permutations that can be formed if there are no restrictions.
From part c), it can be seen that there are $\left(\frac{10!}{4!4!}\right) = 6\ 300$ permutations that can be formed if the 2 P's must be together.

The difference between these two numbers will be how many permutations there are if the 2 P's must never be together as all the options where the 2 P's are together will be taken away from the total number of options.

$$34\ 650 - 6\ 300 = 28\ 350$$

$\therefore$ There are 28 350 permutations that can be made if the 2 P's are not together.

e) the permutation must be a palindrome? That is, the permutation must read the same when read frontwards and backwards. *a toyota* is an example of a palindrome (the space is not considered part of the permutation).

Solution
For the permutation to be a palindrome, the M must be placed directly in the middle as there is only 1 M.

— — — — — M — — — — —

Since the same letters must symmetrical on both sides, the spaces to the left of the M are the only ones that need to be considered.

There are 5 spaces to fill with $\frac{4}{2} = 2$ I's, $\frac{4}{2} = 2$ S's and $\frac{2}{2} = 1$ P.
There are $\binom{5}{2}$ ways to place the 2 I's.
There are now $\binom{5-2}{2} = \binom{3}{2}$ ways to place the 2 S's.
There is now 1 way to place the 1 P in the remaining 1 spot.

Using the product rule:

$$\begin{aligned}\binom{5}{2} \times \binom{3}{2} \times 1 &= \left(\frac{5!}{2!3!}\right) \times \left(\frac{3!}{2!1!}\right) \\ &= \left(\frac{5!}{2!2!}\right) \\ &= 30\end{aligned}$$

$\therefore$ There are $\left(\frac{5!}{2!2!}\right) = 30$ permutations that can be made if the permutations are also palindromes.

5. Anna Lize has seven loonies and wants to distribute them among her three friends. In how many ways can she do this if:

a) it is acceptable for one or more of her friends get nothing.

Solution
There are three possible cases that must be considered: the coins are all given to one friend, the coins are split between two friends, the coins are split between all three friends.

Looking at the first case where the coins are all given to one friend:
There are $\binom{3}{1} = 3$ ways to pick the friend who gets all the coins.

Looking at the second case where the coins are split between two friends:
There are $\binom{3}{2}$ ways to pick the two friends who get the coins. To ensure this case is disjoint from the previous case each of these two must be given at least one coin, so we are now left with distributing the remaining five coins among the two friends. This can be done using the combinations $\{0, 5\}$, $\{1, 4\}$, and $\{2, 3\}$, each of which have $2!$ permutations.
Using the product rule:

$$\binom{3}{2} \times 3 \times 2! \quad = \quad 18 \text{ ways to split the coins among two friends.}$$

Looking at the third case where the coins are split between all three friends:
There are $\binom{3}{3}$ ways to pick the three friends who get the coins. To ensure this case is disjoint from the previous two cases each friend must be given at least one coin, so we are now left with distributing the 4 remaining coins among the three friends. This can be done using the combinations $\{0, 1, 3\}$, $\{0, 2, 2\}$, $\{1, 1, 2\}$, and $\{0, 0, 4\}$, the first of which has 3! permutations and the latter three each have $\frac{3!}{2!}$ permutations.

Using the product rule:

$$\binom{3}{3} \times \left(1 \times 3! + 3 \times \frac{3!}{2!}\right) \quad = \quad 15 \text{ ways to split the coins among three friends.}$$

$\therefore$ Using the sum rule there are $3 + 18 + 15 = 36$ ways Anna can distribute the loonies to her three friends if some of them may get nothing.

b) each friends gets at least 1 loonie.

Solution
Notice that this question is simply the third case from part a),

$\therefore$ (From above) there are 15 ways Anna can split her loonies among three friends if each friend is to receive at least 1 loonie.

c) her best friend, Ann Alysis, gets at least half of the coins and the other 2 friends gets at least 1 loonie.

Solution
If Ann is to get at least half of the loonies, she will receive at minimum 4 loonies. After giving her these 4 loonies and each of the other 2 friends 1 loonie, there is 1 loonie left to give away, and it can be placed in $\binom{3}{1} = 3$ ways.

$\therefore$ There are 3 ways Anna can distribute her loonies if Ann Alysis gets at least half of the coins and the other 2 friends gets at least 1 loonie.