Intermediate Math Circles
Problem Set Solutions
October 19, 2011

1. Three bakers: George, Dameon and Lisa have 1.5 hours to make a wedding cake. It usually takes George, Dameon and Lisa 3 hours, 5 hours and 7 hours respectively to complete a wedding cake on their own. If they work together, will they have it done in time?

   George can make $\frac{1}{3}$ of a cake in 1 hour.
   Dameon can make $\frac{1}{5}$ of a cake in 1 hour.
   Lisa can make $\frac{1}{7}$ of a cake in 1 hour.

   Together in one hour they can make $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}=\frac{35+21+15}{105}=\frac{71}{105}$ of a cake.
   Let t be the time, in hours, required to make a whole cake. Then $\frac{71}{105}:1=1:t$. From this, $t=\frac{105}{71}\doteq 1.48$ hours.

   Since this is less than 1.5 hours, the three working together will be able to make the wedding cake.

2. Carly is knitting a scarf. She can complete a scarf in 6 hours, but is interrupted when she is two thirds of the way done. If Melissa, who takes 8 hours to complete a whole scarf, finishes knitting the scarf what was the total amount of time it took to complete the scarf?

   Since Carly has $\frac{2}{3}$ of the scarf done, she has worked $\frac{2}{3}$ of 6 h = 4 h.

   Let t be the time, in hours, that Melissa needs to finish the scarf. Melissa can knit $\frac{1}{8}$ of a scarf in 1 hour. So to knit the remaining $\frac{1}{3}$ of the scarf:

$$\begin{aligned}\frac{\frac{1}{8}}{1\text{ h}} &= \frac{\frac{1}{3}}{t\text{ h}}\\ \frac{1}{8}t &= \frac{1}{3}\\ t &= \frac{8}{3}=2\frac{2}{3}\text{ h}\end{aligned}$$

   It will take a total of $4+2\frac{2}{3}$ h or 6 hours and 40 minutes to complete the scarf.

3. If a car travels at a constant speed of $60\ \frac{\text{km}}{\text{h}}$ for 5 s, how many metres does it travel?

   Distance = Speed × Time $= 60\ \frac{\text{km}}{\text{h}}\times 5\text{ s}\times\frac{1000\text{ m}}{1\text{ km}}\times\frac{1\text{ h}}{60\text{ min}}\times\frac{1\text{ min}}{60\text{ s}}=\frac{60\times5\times1000}{60\times60}\text{ m}=\frac{250}{3}\text{ m}.$

   $\therefore$ the car travels $\frac{250}{3}$ metres, approximately 83 metres.

4. A car travels 800 m at a constant speed of 80 $\frac{\text{km}}{\text{h}}$. Determine the length of time, in seconds, it took the car to travel 800 m.

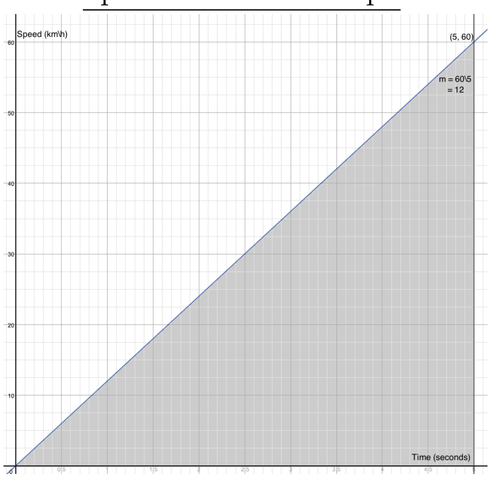
$$\begin{aligned}
800m &= 80\ \frac{\text{km}}{\text{h}} \times t\ \text{s} \\
800m &= 80\ \frac{\text{km}}{\text{h}} \times t\ \text{s} \times \frac{1000\ \text{m}}{1\ \text{km}} \times \frac{1\ \text{h}}{60\ \text{min}} \times \frac{1\ \text{min}}{60\ \text{s}} \\
800 &= \frac{80(1000)t}{60(60)} \\
t &= \frac{800(60)(60)}{80(1000)} \\
t &= 36\ \text{s}
\end{aligned}$$

$\therefore$ it took the car 36 seconds to travel 800 metres.

5. If a car accelerates at a constant rate from 0 $\frac{\text{km}}{\text{h}}$ to 60 $\frac{\text{km}}{\text{h}}$ in 5 seconds, what distance does it travel in this time?

$$\begin{aligned}
\text{Acceleration} &= \frac{\text{Change in Speed}}{\text{Time}} \\
&= \frac{60\ \frac{\text{km}}{\text{h}} - 0\ \frac{\text{km}}{\text{h}}}{5\ \text{s}} \\
&= 12\ \frac{\text{km}}{(\text{h})(\text{s})}
\end{aligned}$$

Speed Vs. Time Graph

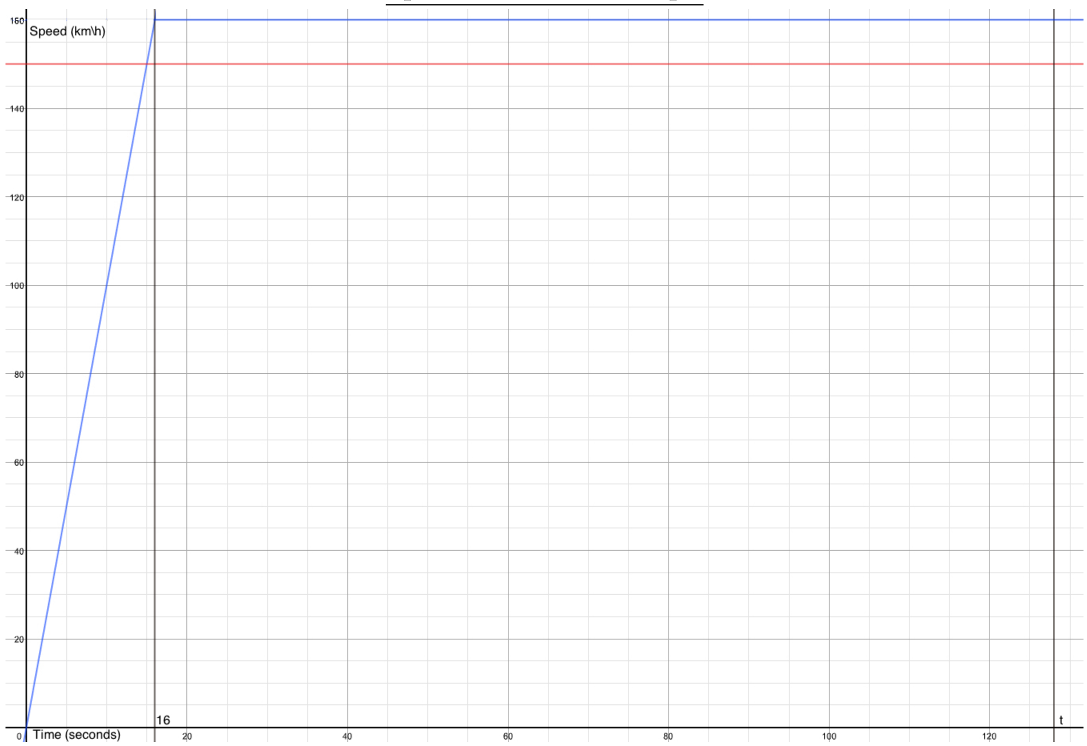


$$\begin{aligned}
\text{Distance} &= \text{Area of Triangle} \\
&= \frac{1}{2}(5\ \text{s})\left(60\ \frac{\text{km}}{\text{h}}\right) \\
&= \frac{1}{2}\ (5\ \text{s})\left(60\ \frac{\text{km}}{\text{h}}\right)\left(\frac{1\ \text{h}}{60\ \text{min}}\right)\left(\frac{1\ \text{min}}{60\ \text{s}}\right)\left(\frac{1000\ \text{m}}{1\ \text{km}}\right) \\
&= \frac{1}{2}\left(\frac{300000}{3600}\right)\ \text{m} \\
&= \frac{250}{6}\ \text{m} \\
&= \frac{125}{3}\ \text{m}
\end{aligned}$$

$\therefore$ the car travels $\frac{125}{3}$ metres or approximately 41.7 metres in 5 seconds.

6. A car travelling at a constant speed of 150 $\frac{\text{km}}{\text{h}}$ passes a police cruiser that is stopped on the side of the road. The police cruiser accelerates at a uniform rate of 10 $\frac{\text{km}}{\text{h}}$ per second to a top speed of 160 $\frac{\text{km}}{\text{h}}$ and chases the car. At what time after the car passed the police cruiser does the police cruiser catch up to the car?

Speed Vs. Time Graph



Let t be the time it takes for the cruiser to travel the same distance as the car. (The area under the curve is the distance travelled.)

In t seconds, the car travels:

$$d = \left(150\ \frac{\text{km}}{\text{h}}\right)(t\ \text{s})$$

The cruiser travels:

$$d = \frac{1}{2}\left(160\ \frac{\text{km}}{\text{h}}\right)(16\ \text{s}) + \left(160\ \frac{\text{km}}{\text{h}}\right)((t-16)\ \text{s})$$

$$\begin{aligned}
\therefore \qquad 150t\left(\frac{\text{km}\times\text{s}}{\text{h}}\right) &= 1280\left(\frac{\text{km}\times\text{s}}{\text{h}}\right) + 160\,(t-16)\left(\frac{\text{km}\times\text{s}}{\text{h}}\right) \\
150t &= 1280 + 160t - 2560 \\
-10t &= -1280 \\
t &= 128 \text{ seconds} \\
&= 2 \text{ minutes } 8 \text{ seconds}
\end{aligned}$$

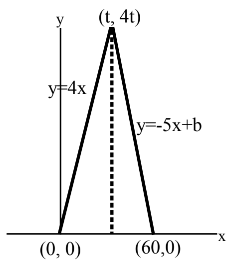
$\therefore$ it takes the cruiser 2 minutes and 8 seconds to travel the same distance as the car.

7. **Challenge 1:** A car accelerates at a uniform rate of 4 $\frac{\text{m}}{\text{s}^2}$ and brakes at a uniform acceleration rate of -5 $\frac{\text{m}}{\text{s}^2}$. If the car begins and ends at a stop and travels for 60 seconds, what is the maximum speed it can reach, and what distance will it cover?

   Acceleration is the slope of the line.

   The equation of the line with slope 4 that passes through $(0, 0)$ is $y = 4x$.

   Similarily, the equation of the line with slope -5 is $y = -5x + b$.



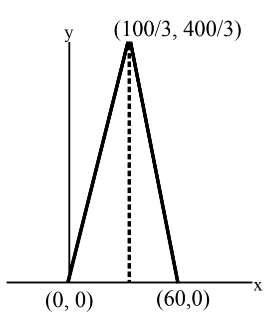
   Substituting the point $(t, 4t)$ into the equation $y = -5x + b$:

$$\begin{aligned} 4t &= -5t + b \\ 9t &= b \\ \therefore y &= -5x + 9t \end{aligned}$$

   Substituting the point $(60, 0)$ into the equation above:

$$\begin{aligned} 0 &= -5(60) + 9t \\ 0 &= -300 + 9t \\ 300 &= 9t \\ \frac{100}{3} &= t \\ 4t &= 4\left(\frac{100}{3}\right) \\ 4t &= \frac{400}{3} \\ \therefore \text{The point of intersection is } (t, 4t) &= \left(\frac{100}{3}, \frac{400}{3}\right) \end{aligned}$$

   Solving for the area under the triangle:



$$\begin{aligned} \text{Area} &= \text{Base} \times \text{Height} \\ &= \frac{1}{2}(60)\left(\frac{400}{3}\right) \\ &= 4\ 000 \end{aligned}$$

   $\therefore$ the maximum speed the car can reach is $\frac{400}{3}$ m/s and in total, it covers 4 000 m.

8. **Challenge 2:** If a car can accelerate at a constant rate of 3 $\frac{\text{m}}{\text{s}^2}$ and can brake at a constant acceleration rate of $-5\ \frac{\text{m}}{\text{s}^2}$, what is the maximum speed it can reach over a distance of 500 metres if it begins and ends at a stop?

   Acceleration is the slope of the line.

   The equation of the line with slope 3 that passes through (0, 0) is $y = 3x$.

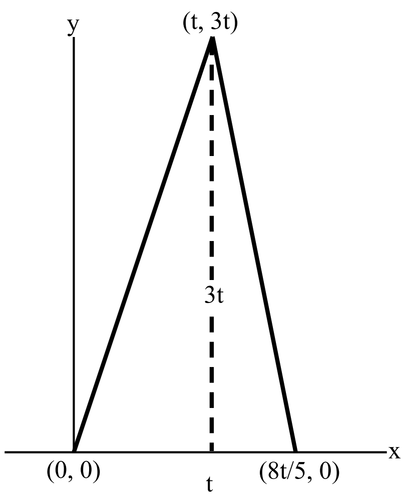
   Let the lines intersect at time t seconds. Since the point of intersection is on the first line, it can be written $(t, 3t)$.

   The second line has slope -5 and also passes through the point $(t, 3t)$. Its equation can now be determined:

$$\begin{aligned} y - 3t &= -5(x - t) \\ y - 3t &= -5x + 5t \\ y &= -5x + 8t \end{aligned}$$

   Setting $y = 0$ and solving for the x-intercept:

$$\begin{aligned} 0 &= -5x + 8t \\ 5x &= 8t \\ x &= \frac{8}{5}t \end{aligned}$$



$$\begin{aligned} \text{Area} &= \frac{1}{2} \times \text{Base} \times \text{Height} \\ 500 &= \frac{1}{2}\left(\frac{8t}{5}\right)(3t) \\ 500 &= \frac{24t^2}{10} \\ 5\ 000 &= 24t^2 \\ \frac{5\ 000}{24} &= t^2 \\ \frac{625}{3} &= t^2 \end{aligned}$$

   Because time is positive, we can discard the negative root:

$$\begin{aligned} t &= \frac{25}{\sqrt{3}}\ \text{s} \\ &= \frac{25\sqrt{3}}{3}\ \text{s} \end{aligned}$$

   The car reaches it's maximum speed at the point $(t, 3t)$:

$$\begin{aligned} 3t &= 3 \times \frac{25\sqrt{3}}{3} \\ &= 25\sqrt{3} \end{aligned}$$

   $\therefore$ The car reaches has a maximum speed of $25\sqrt{3}$ m/s.