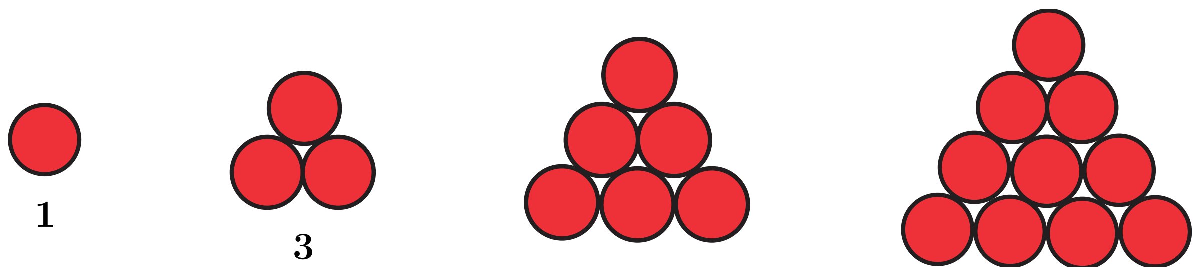
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7 & 8 Math Circles
November 16, 2011
Polygonal Numbers

What is a Triangular Number?

A *Triangular Number* is a number that can be arranged into an equilateral triangle, as shown below:



1   3   6   10

Looking at the first four triangular numbers, can you find the pattern to guess what the next triangular number is?

Triangular numbers are given a special notation T_n for any positive integer n (*Example:* $T_1 = 1$, $T_2 = 3$ and $T_3 = 6$). We also say that the *triangular root* of 3 is 2, the triangular root of 6 is 3, etc. (*Note:* Do not confuse this as the cubed root, they are two different things!)

The Formula

Looking at the above diagrams we can formulate an equation for solving for every triangular number:

$$T_n = \sum_{i=1}^{n} n$$

Consider This...

You were talking during your math class and as a punishment, your teacher made you sum all the numbers from 1 to 100 together without your calculator. After writing the first ten numbers, you think to yourself that there must be a better way to do this.

1 2 3 4 5 6 7 8 9 10 ... 96 97 98 99 100

Instead, you try to sum the larger numbers first, starting with 100 and working your way down.

100 99 98 97 96 ... 10 9 8 7 6 5 4 3 2 1

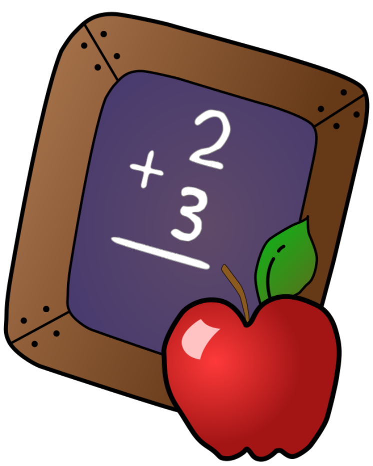
This becomes very tedious after a while as well, so you sit there wondering what you could do next to make it easier. Out of boredom, you begin adding the numbers in the first column together and then the second column, and so on. You notice something very strange and interesting.

Each column sums to 101!

Since there is 100 columns, and each column sums to 101, the total sum of the two columns is 100×101. You also know that the sum of each row is equal, which means to find the sum of the numbers from 1 to 100 you just have to perform the following calculation:

$$\frac{100 \times 101}{2} = 5050$$

You have finished your punishment in less than 5 minutes, and your teacher is amazed!



Interesting...

Legend has it that this is how this formula was created. The famous mathematician Gauss, as a punishment in school, was required to sum all these digits. Instead of being punished, Gauss created a famous formula!

Generalizing the Formula

The formula that Guass discovered can be generalized into the form:

$$T_n = \sum_{i=1}^{n} n = \frac{n(n+1)}{2}$$

Example Set 1

1. Determine the value of each of the following:
   (a) T_{77}
   (b) T_{500}
   (c) The 400$^{\text{th}}$ triangular number
   (d) The 245$^{\text{th}}$ triangular number
2. Using the same method as Gauss (and a little extra thinking), how would you sum the numbers from 19 to 99?
3. Looking at the first few triangular numbers, what can you say about the sum of consecutive triangular numbers?
4. Notice how the first two triangular numbers are odd, while the next two are even. Does this pattern hold true for all the triangular numbers? Prove your answer.
5. You went to a camp this summer. There were 25 people at the camp. On the last day when everyone was leaving, everyone got a hug from every other person. How many hugs were there all together?

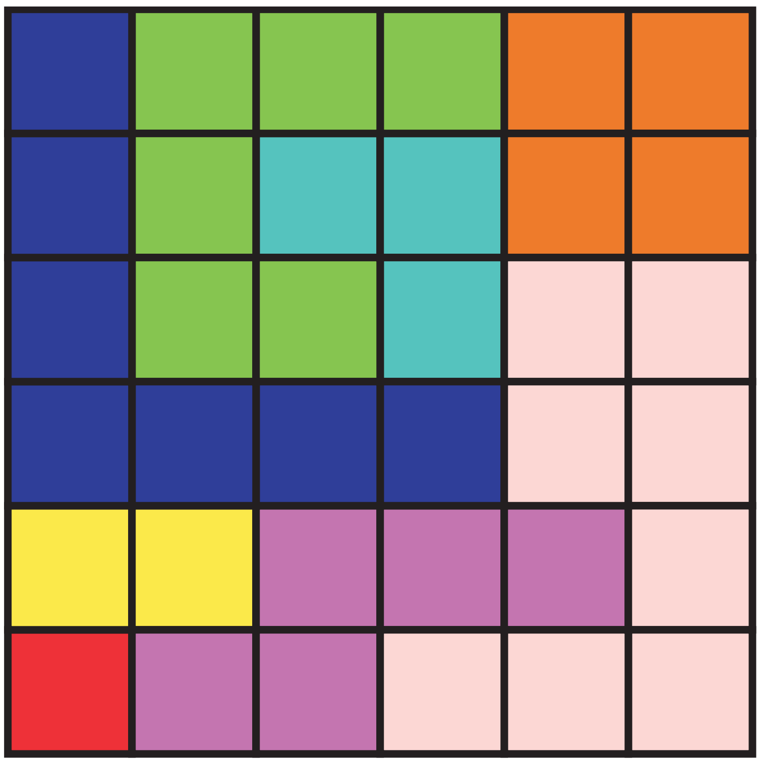
6. The grocery store is making a triangular pyramid out of soup cans in the middle of the store. If the manager has 65 cans to use, how many of these cans will be needed to make the pyramid and how many levels high will it be (*Assumption:* All the soup cans are the same size)?

Rickie Puzzlers

A mathematicians named Rickie Chase, developed a puzzle using a few facts:

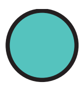
- Some triangular numbers are also square numbers (for example, $6 \times 6 = 36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$)
- Pieces could be made with areas of 1, 2, 3, 4, 5, 6, 7 and 8 and if made correctly, these pieces could successfully fit into a 6×6 square

Example:



Pentagonal Numbers

Similar to triangular numbers, pentagonal numbers are numbers that can be arranged into a regular pentagon as shown below:



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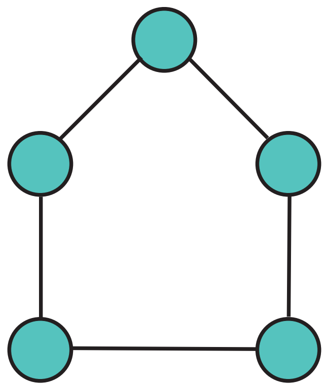

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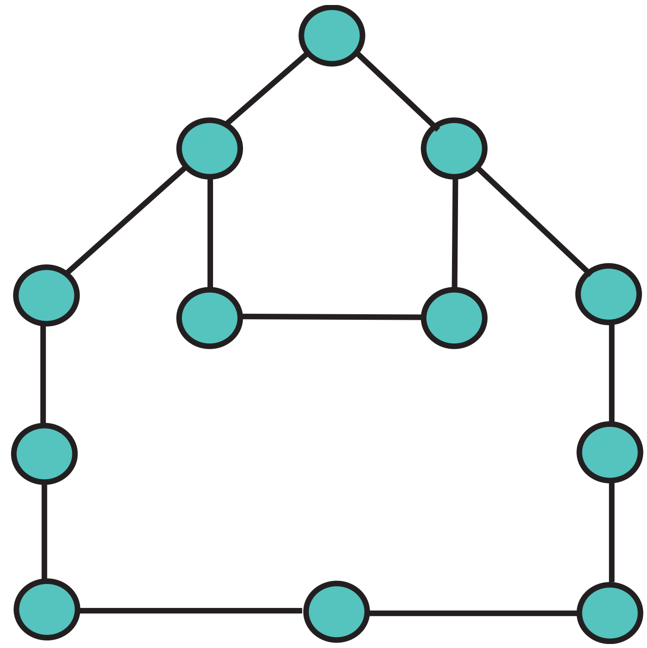

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Interesting...
There are other numbers such as square (we are familiar with these), pentagonal, hexagonal, heptagonal and octogonal numbers!
These are called polygonal numbers.

The Formula

Interestingly, the formula for each polygonal set of numbers is based on the formula for triangular numbers.

Looking at the sequence of triangular numbers, we have:

$$1 \ , \ 3 \ , \ 6 \ , \ 10 \ , \ 15 \ , \ 21 \ , \ldots$$

As we already know, these can be written as:

$$T_{n-1} + n = T_n$$

We will now find a sequence of terms using the formula:

$$2T_{n-1} + n$$

this gives us the sequence:

$$1 \;,\; 4 \;,\; 9 \;,\; 16 \;,\; 25 \;,\ldots$$

Which we know are all square numbers!

Now we will find a sequence of terms using the fomula:

$$3T_{n-1} + n$$

What can be said about this sequence?

Generalizing the Formulas

Looking at the formula we found for square numbers:

$$2T_{n-1} + n$$

We know that $T_{n-1} = \frac{(n-1)n}{2}$ from our generalization of the triangular number sequence which allows us to simplify the formula above to:

$$2 \times \tfrac{(n-1)n}{2} + n$$

$$= (n-1)n + n$$

$$= n^2 - n + n$$

$$= n^2$$

This is the explicit formula that we are familar with!

Example Set 2

1. Find the first 10 terms of each sequence of polygonal numbers for each of the following:
   (a) Hexagonal Numbers (6 edges)
   (b) Heptagonal Numbers (7 edges)
   (c) Octagonal Numbers (8 edges)
2. Using Question 1, find the pattern between succeeding polynomial numbers.
3. Find the explicit formula for the sequence of polygonal numbers (by using the explicit triangular formula) for each of the following:
   (a) Pentagonal Numbers(5 edges)
   (b) Hexagonal Numbers (6 edges)
   (c) Heptagonal Numbers (7 edges)
   (d) Octagonal Numbers (8 edges)