University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7 & 8 Math Circles
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Prime Numbers

Factors

Definition: A *factor* of a number is a whole number that divides evenly into the other number. So, when you divide by the factor the remainder is zero.

Factors come in pairs. When you divide a number by one of its factors, the answer or quotient is the paired factor.

Is 8 a factor of 32?

Yes, 32 divided by 8 gives us an answer of 4 with no remainder.

Is 4 a factor of 47?

No, 47 divided by 4 gives us an answer of 11 with a remainder of 3.

Example 1 State all the factors of the following numbers.

1) 50 2) 18 3) 29 4) 36

Notice that 36 has an odd number of factors. Each factor still has a pair, but 6 pairs with itself, and is only listed once.

Prime Numbers

Definitions:
A *prime number* is a number that has only two factors: 1 and itself.

A *composite number* it is a number with factors other than 1 and itself.

Exceptional Case: The number 1 is neither prime nor composite by definition.

Example 2 Determine whether the following numbers are prime or composite. If it is composite, list out its factors.

(a) 23

(b) 38

(c) 53

(d) 51

Are there any even numbers that are prime?

Yes, 2 is the only prime number that is even. Think about why.

How can we find prime numbers?

We can find them using the method called **Sieve of Eratosthenes**.

Instructions:

1. Cross off 1 since it is not prime.
2. Find the first prime number (2) and circle it.
3. Go through the rest of the chart and cross off all of the multiples of two.
4. Go back to the beginning of the chart and find the first number that is not crossed off. Circle it, it is prime. Then go through the rest of the chart, crossing off the multiples of this number.
5. Repeat step 4, until every number in the chart is either circled or crossed off.
6. The numbers that are circled are the prime numbers.

Example 3 Use this method to find all the prime numbers from 1 to 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

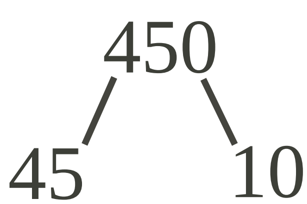
Write down the primes less than 100. There should be 25 of them.

Prime Factorization

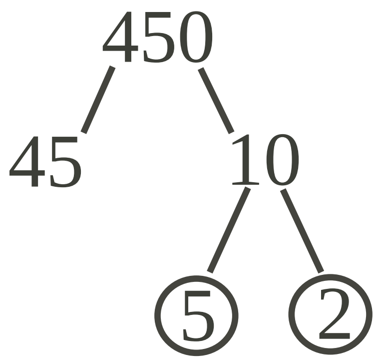
One way of describing numbers is by breaking them down into a product of their prime factors. This is called prime factorization. Every positive number can be prime factored. By definition the prime factorization of a prime number is the number itself, and the prime factorization of 1 is 1.

Let's prime factor 450 using a factor tree.

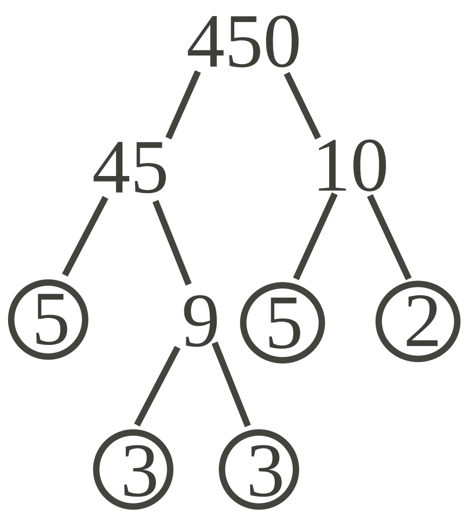
Solution First think of a factor of 450. We don't want to use 1 or 450.
Let's use 10.
Now what factor is paired with 10? That factor is 45, since $10 \times 45 = 450$.



Is 10 prime? No, so we have to prime factor 10.
What is a pair of factors of 10? A pair is 2 and 5.
Both 2 and 5 are prime, so we are done this part.



Is 45 prime? No, so we have to prime factor it.
What is a pair of factors? One pair is 9 and 5.
The number 5 is prime, but 9 has factors 3 and 3.



We have decomposed 450 into all prime numbers, so we can now write it as a product of these primes.

$$450 = 2 \times 3 \times 3 \times 5 \times 5 = 2 \times 3^2 \times 5^2$$

Note: Prime factorizations are unique. Every number has only one prime factorization, and no two numbers share the same prime factorization.

Greatest Common Divisor

Definition: The *greatest common divisor* or *gcd* of two or more numbers is the largest number that is a factor of each of the numbers. This can also be called the greatest common factor.

Example 4 Find the greatest common divisor of 30 and 24.

30 has factors:
24 has factors:
The factors that both numbers have in common are:
Of these, ____ is the largest so the greatest common divisor of 30 and 24 is ____.

Example 5 Find the greatest common divisor of 546 and 840. Use factor trees to help you.

To find the greatest common divisor of these numbers, we look at all of the prime factors they have in common.

The common prime factors of 546 and 840 are:
Therefore, their greatest common divisor is:

Least Common Multiple

Definitions:
A *multiple* of a number is the product of the number and another whole number. Factors and multiples are closely related. For example, 9 is a factor of 90, so 90 is a multiple of 9.

The *least common multiple* or *lcm* of two or more numbers is the smallest number that is a multiple of each of the numbers.

One method of finding the least common multiple involves listing the multiples of each number until you find the first number that is a multiple of both.

Example 6 Find the least common multiple of 6 and 8.

The first few multiples of 6 are:
The first few multiples of 8 are:
The first multiple that occurs in both is ____, so the least common multiple of 6 and 8 is ____.

This method may take too long if we are dealing with larger numbers. Instead, we will use another method involving prime factors.

Example 7 Find the least common multiple of 80 and 144. Use factor trees to help you.

The least common multiple must be the product of each prime factor with the greatest power.
Therefore, the least common multiple of 80 and 144 is:

Example 8 Complete the following chart.

A	B	C	D	E	F
First Number	Second Number	gcd of A and B	lcm of A and B	$A \times B$	$E \div C$
495	945				
168	234				
520	189				
345	765				

(a) What do you notice about the relationship between column D and column F?

(b) Write this relationship in the form of an equation.

(c) Why do you think this equation is true?

Exercises

1. Find a prime factorization of 24840
2. Find two different factor trees for 150. What do you notice about their prime factorizations?
3. What is the smallest prime that is greater than 200?
4. Colin splits his collection of 36 baseball cards into groups, so that no extra cards remain. What are the possible ways he split up his cards?
5. A rectangular box has an area of $7644cm^2$. What are the possible dimensions for the box?
6. What is the smallest number that we can multiply 1512 by to get a perfect square?
7. Find the greatest common divisor and the least common multiple of 168 and 420.
8. A florist has 72 roses, 84 tulips and 48 orchids that she wants to use to create bouquets. What is the largest number of identical bouquets she can put together without having any flowers left over?
9. Sam was buying hot dogs and hot dog buns for a backyard barbeque. Hot dogs come in packs of 16, but buns come in packs of 12. How many packs of each will Sam have to buy so that there are no hot dogs or buns left over?
10. Two flashing signs are turned on at the same time. One sign flashes every 4 seconds and the other flashes every 6 seconds. How many times will they flash at the same time in 1 minute?

11. Find two numbers who have a product of 81 and a sum of 30.
12. Two numbers have a sum of 300. What are all of the possible sums of these numbers?
13. Kevin the baker purchases apples to bake into pies. If he can put 7, or 8, or 10 apples in each pie and be left with 100 apples, what is the least number of apples Kevin must have purchased?
14. I am telling my friend a riddle. I tell her that the product of my three siblings' ages is 72, and the sum of their ages is the day of the month on which her birthday falls. After a bit of thinking, she tells me I haven't given her enough information. I tell her that my oldest sibling doesn't like chocolate. How old are my three siblings?