University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7 & 8 Math Circles
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Fractals

Definition: A ***fractal*** is a geometric figure that is formed by a repeated application of a certain process (iteration). Fractals do not change complexity at any level of magnification.

What does it all mean?

Basically, a fractal is a kind of shape that looks approximately the same no matter how much we zoom in or out on it.

To construct a fractal, we use a tool called "*iteration*". This means applying a step to a original figure (usually a simple shape), taking the result, and applying the step again. With fractals we repeat this process infinitely many times (it never ends).

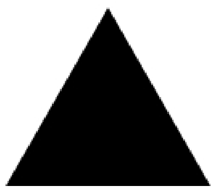
Example 1
We will create a fractal called the *Sierpinski Triangle* by following these steps:

1. Draw an equilateral triangle
2. Connect the midpoints of all three sides
3. Remove (by shading in or cutting out) the resulting upside-down middle triangle
4. Repeat steps 2-3 with the three smaller triangles that were made
5. Repeat step 4 infinitely many times!

Let's look into this shape: What shape do we find if we look at the bottom right corner of Sierpinski's Triangle? We find Sierpinski's Triangle! And if we were to focus on the corner of this corner, we would find Sierpinski's Triangle again!
This property is called "*self similarity*", and it must be present in all fractals.

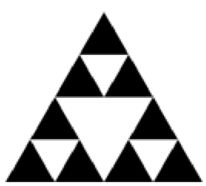
Definition: An object is ***self similar*** if parts of that object look similar to the whole object.

An interesting property of Sierpinski's Triangle
What is its area?

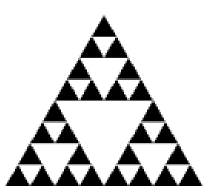


At step 1 we have a simple equilateral triangle, we will say it has an area of 1.

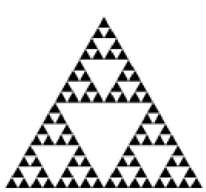
At the first iteration, we remove one quarter of the original shape and have three quarters left, so our area is now $\frac{3}{4}$



At the second iteration, we remove 3 out of the 12 smaller triangles left after the first iteration, and so are left with $\frac{9}{12} = \frac{3}{4}$ of the first iteration's triangle. This means we now have $\frac{3}{4} \times \frac{3}{4} = (\frac{3}{4})^2 = 0.5625$ of our original triangle left.



If we continue this pattern, we see that at the end of each iteration, we are left with $\frac{3}{4}$ of what we had at the beginning of that iteration.
In relation to the original triangle, after iteration n, our area is $(\frac{3}{4})^n$.

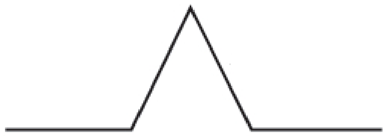


After an infinite number of iterations, the area of the triangle is 0.

Sierpinski's Triangle is an example of a "*base-motif*" fractal, where each occurrence of a certain shape (the "*base*") is replaced by a slightly modified version of itself (the "*motif*").
In the case of Sierpinski, the base is a triangle and the motif is three smaller triangles.
There are many, many different base-motif fractals...

Example 2
Follow these steps to build the famous *Koch Snowflake*:

1. Draw an equilateral triangle
2. Replace each line segment (these are our bases) with the following motif:

   Make sure your points are **outside** the original triangle
3. Repeat step 2 with every line segment, both the newly created ones and those left from the previous iteration
4. Repeat step 3 infinitely many times!

Notice the self similarity.

Questions

1. What is the perimeter of this shape?
2. What about the area?

More Examples of Base-Motif Fractals

3. Box Fractal

1. Draw a filled-in square
2. Divide the square into 9 smaller squares
3. Remove the centre square from each side
4. Repeat steps 2-3 with each of the 5 remaining squares
5. Repeat step 4 infinitely many times!

4. H-Fractal

1. Draw a horizontal line
2. At both ends of this line, add a perpendicular line that is half as long and intercepts at its midpoint
3. Repeat step 2 for both of the newly added lines
4. Repeat step 3 infinitely many times!

5. Circle Fractal

1. Draw a circle
2. Inside of your circle, draw two smaller circles that have half the diameter, they should be able to fit snugly side by side in the middle of the larger circle
3. repeat step 2 with each smaller circle
4. repeat step 3 an infinite amount of times!

Which of these fractals has an infinite perimeter and an area of 0?

Applications

So fractals are pretty to look at, but when do we see them in the real world??

Fractals are **ALL OVER THE PLACE:**
You can find them
- In the sky: In lightning strikes and cloud formations
- In the ground: In mountain ranges, shorelines, and cracked earth
- In the water: In oil spills, seashells, and coral reefs
- In the trees: In branches, leaves, and ferns
- In technology: In cooling circuits, and cell phone antennae
- In YOUR BODY: In blood vessels, neurons, and bronchial tubes

You can see them in artwork and special effects, hear them in fractal music arrangements, and even *taste* them with foods like broccoflower.

Fractals have even been where no human has gone before...

Example 6
Use the following steps to construct a universally famous fractal called the *Cantor Set*:

1. Draw a horizontal line
2. Directly underneath your line, draw the exact same line, but with the middle piece removed
3. Repeat step 2 with both of your newly created lines
4. Repeat step 3 infinitely many times!

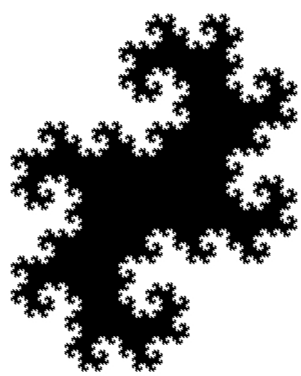
Why is this fractal universally famous?
If you put circles through the points in your last iteration, you will have a simple model of Saturn's ring, which is actually composed of many many self similar rings.

There are even fractals we can hold in our hands...

Example 7
Create a *Dragon Curve* using a strip of paper and the following steps:

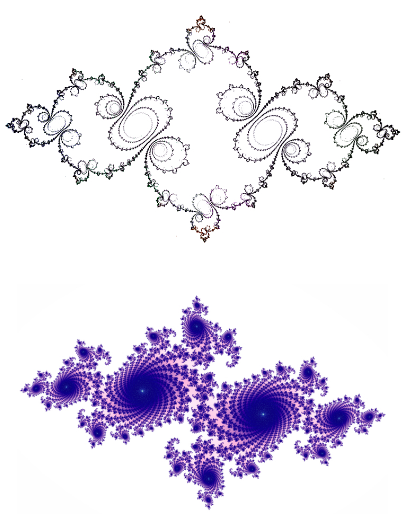
1. Fold your strip of paper in half once. Unfold it to make a right angle, this is your first iteration
2. fold your paper back in half and fold it in half once more **in the same direction**. That is, if you folded your paper from right to left to start, fold it from right to left again. Unfold and make sure all corners are right angles. You will see that our new fractal is made up of two copies or our previous iteration stuck together at a right angle
3. Repeat step 2 until your paper becomes too difficult to fold (about two or three more iterations). Be sure to unfold and take note of the self similarity after each iteration

If we were able to fold the paper an infinite amount of times (we can use a computer to simulate this process), The result would be the following figure:

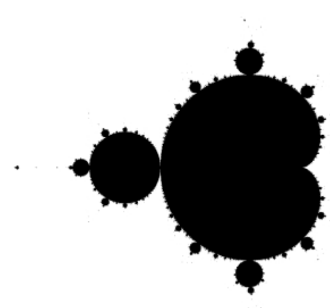


The more difficult we make our iterations, the more beautiful and intricate our figures become. Here are two examples of fractals that use a higher level math concept called complex numbers:

The Julia Set



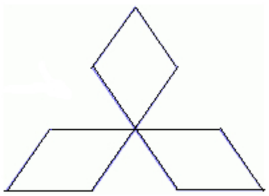
The Mandelbrot Set



To see the Mandelbrot Set in amazing detail,
check out the Mandelbrot zoom video at
http://www.youtube.com/watch?v=foxD6ZQlnlU

Exercises

1. Koch Antisnowflake
What happens to the Koch Snowflake if we draw our points inside the triangle instead of outside? Here is the first iteration:



Continue with the iterations to construct the Koch Antisnowflake.

2. Cesaro

1. Draw a filled in square
2. Remove a sharp spike from the centre of each side, resulting in the following motif:

3. Repeat step 2 with each of the 4 square-like sections of the motif
4. Repeat step 3 an infinite amount of times!

3. Serpinski Carpet

1. Draw a square
2. Divide the square into 9 smaller squares, fill in the middle square and erase any lines remaining from dividing the square
3. Repeat step 2 with the 8 remaining squares
4. Repeat step 3 infinitely many times!

Note: This fractal can be extended to 3 dimensions. Divide a cube into 27 smaller cubes, then remove the centre cube and the cubes at the centre of each side. If you repeat this process with the 20 remaining cubes, you will end up with a 3D fractal called the "Menger Sponge".

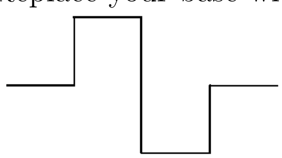
4. Pythagoras Tree

1. Draw a simple "house" shape (a square with a 45-45-90 degree triangle on top)
2. Add another house shape on top of both sides of your previous house
3. Repeat step 2 for the two newly made house shapes
4. Repeat step 3 infinitely many times!

What happens if we use a 30-60-90 degree triangle instead? Try doing this and:
a) orient the triangle the same way for every step
b) alternate which way the triangle will face for each iteration

5. Minkowski Sausage

1. Draw a straight line. This is your base
2. Replace your base with the following motif:



3. Repeat step 2 for all newly created line segments
4. Repeat step 3 an infinite amount of times!

6. Star of David Fractal

1. Draw an equilateral triangle
2. Draw an inverted triangle directly on top of the previous triangle (ie. make a Star of David)
3. Repeat step 2 for all six of the smaller triangles you have just created
4. Repeat step 3 an infinite amount of times!

Can you invent any other fractals?

Extra Resources

Slow-motion fractal lightning: http://picsthatdontsuck.com/img/slow-motion-lightning.gif

Fractal Music
http://www.fractovia.org/art/fmusic/voicez_128k.mp3
http://www.fractovia.org/art/fmusic/dexia_128k.mp3
http://www.fractovia.org/art/fmusic/eyesight_128k.mp3
http://www.brotherstechnology.com/audio/stretching-out.html
http://www.fractovia.org/art/fmusic/trabalg_128k.mp3

Fractal Cut-outs
http://fractalfoundation.org/resources/fractivities/fractal-cutout/
mathinscience.info/public/fractal_cards/fractal_cuts.pdf