University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7 & 8 Math Circles
November 23, 2011
Jeopardy

Round 1

Arithmetic

1.

$$\begin{aligned}(-10) + (-4) + (\frac{6}{2}) \times 3 &= (-14) + 3 \times 3 \\ &= (-14) + 9 \\ &= -5\end{aligned}$$

2. 4 and -4

3.

$$\begin{aligned}\frac{(5^4)^3 \times 5^2}{5^7} &= \frac{5^{12} \times 5^2}{5^7} \\ &= \frac{5^{14}}{5^7} \\ &= 5^7\end{aligned}$$

4. Since we know if any number has a even numbered exponent, the result will be a positive number, and we also know that $R^4 \geq R^2$, then we know the result will be ≥ 0.

5.

$$\frac{7}{4 - \frac{7}{2}} = 14$$

Prime Numbers

1. The *Sieve of Eratosthenes* was used in class to find the prime numbers from 1-100.

2. 1 is considered to be neither prime nor composite.

3. The lowest common multiple (lcm) of 8 and 12 is 24 (this can be found with a factor tree). Therefore we must buy 3 packages of hotdogs and 2 packages of buns.

4.

$$53\ 000 = 2^3 \times 5^3 \times 53$$

5.

$$\begin{aligned} \frac{6\ 320}{395} &= \frac{2^4 \times 5 \times 79}{5 \times 79} \\ &= 2^4 \\ &= 16 \end{aligned}$$

Patterns/Sequences

1. 3, 5, 4, 7, 6, 10, 9, 14, 13
2. A number is divisible by 8 if the last three digits are divisible by 8.
3.

$$\begin{aligned} F_{40} &= \frac{(\frac{1+\sqrt{5}}{2})^{40} - (\frac{1-\sqrt{5}}{2})^{40}}{\sqrt{5}} \\ &= 102\ 334\ 155 \end{aligned}$$

4.

$$\begin{aligned} T_{29} &= \frac{29(29+1)}{2} \\ &= 435 \end{aligned}$$

$$\begin{aligned} T_{30} &= \frac{30(30+1)}{2} \\ &= 465 \end{aligned}$$

$$T_{30} - T_{29} = 30$$

Note: From the definition of triangular numbers, we could just say 30 was the difference as well

5.

$$H_n = n(2n-1)$$

Note: For the proof, refer to November 16 notes.

Modular Arithmetic

1. Answers may vary.
2. MATH CIRCLES
3. $125 \equiv 6 (\text{mod } 7)$
   Therefore in 125 days from now, it will be 6 days after Wednesday, which is a Tuesday.
4. GBLMXOAV
5. While Susan was living, there has been 3 leap years (2008, 2004, and 2000– Note: although she was born in 1996, she was born after the leap year), which means we must add 3 days on to the total.
   Susan has been alive for:

$$\begin{aligned}(365 \times 15) + 3 &= 5\ 478 \\ 5\ 478 &\equiv 4 (\text{mod } 7)\end{aligned}$$

   We must go backwards 4 days from now as we are looking at the past. Therefore Susan was born on a Saturday.

Probability

1. Since $\frac{50}{50}$ of the students are under the age of 10, this means that all the students are under the age of 11. Therefore the probability that the student is under the age of 11 is 100%.
2. The probability of rolling two 6's in a row using a standard fair die is $(\frac{1}{6}) \times (\frac{1}{6}) = \frac{1}{36}$
3. We can draw a table of all the possible sums of two die

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	**7**
2	3	4	5	6	**7**	8
3	4	5	6	**7**	8	9
4	5	6	**7**	8	9	10
5	6	**7**	8	9	10	11
6	**7**	8	9	10	11	12

   We can see that $\frac{6}{36}$ of these sums are equal to 7.

   Therefore the probability of rolling a 7 is $\frac{6}{36} = \frac{1}{6}$.

4. $1\ 000 \times 0.60 \times 0.40 \times 0.75 = 180$
   There are 180 boys under 15 with brown hair.

   Therefore the probability that the person chosen at random is a boy under 15 with brown hair is $\frac{180}{1\ 000} = \frac{9}{50}$.

5. We know that we could not have chosen Bag A, since there are no blue marbles in this bag and we know the first marble we picked out of the bag was blue. Looking at the other options we know:

   If we had chosen Bag B, we would have the following options:

Pick 1	**Pick 2**
Blue 1	Red
Blue 1	Blue 2
Blue 2	Red
Blue 2	Blue 1

   If we had chosen Bag C, we would have the following options:

Pick 1	**Pick 2**
Blue 1	Blue 2
Blue 1	Blue 3
Blue 2	Blue 1
Blue 2	Blue 3
Blue 3	Blue 1
Blue 3	Blue 2

   If we had chosen Bag D, we would have the following options:

Pick 1	**Pick 2**
Blue	Red 1
Blue	Red 2

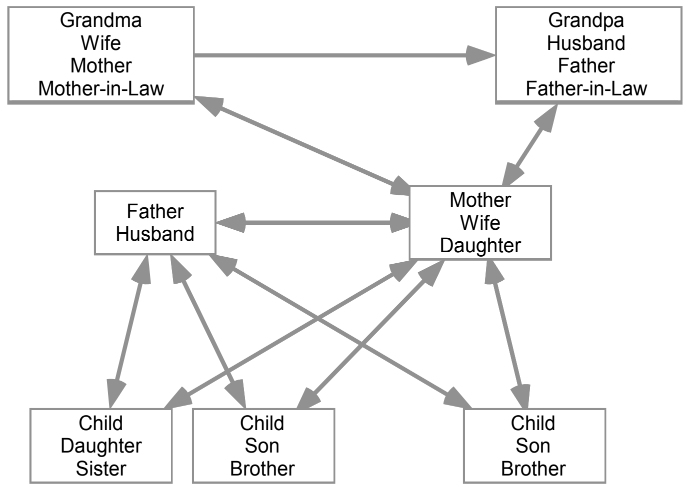
   Looking at the second picks, we see that $\frac{8}{12}$ of the choices are blue marbles. Therefore the probability of picking a second blue marble is $\frac{2}{3}$.

Round 2

Logic

1. The name of the 4^{th} dog is Dabby.

2. There are 7 people at the dinner



3. Let's say Joe was born on December 31^{st}, 2000. If today is January 1^{st} 2013, then Joe turned 12 yesterday. Next year (in 2014), Joe will be turning 14 years old.

4. You can ask: I will be thrown into the fire.
   Looking at this we see that the King cannot say this is true, otherwise he would be thrown into jail. But if the statement is false, the thief would be thrown into the fire, which would mean the statement is true.

5. Even though there are only 62 squares, we notice that however we place the dominoes, half will cover a white square and half will cover a black square. Since you took out two black squares, you will always have two white squares left over no matter how you position the dominoes. Therefore, it is impossible to cover the board with dominoes.

Word Problems

1.

$$\begin{aligned} \frac{87+69+97+80+79+x}{6} &= 85 \\ 412+x &= 510 \\ x &= 98 \end{aligned}$$

Therefore, Joseph must achieve a minimum mark of 98% on the final test.

2. Let A be the age of Audrey now and B be the age of her brother now.
We can make the following equations from the given information:

$$
\begin{aligned}
B+2 &= A \\
\\
A+1 &= 2(B+1) \\
A+1 &= 2B+2 \\
A &= 2B+1
\end{aligned}
$$

Substituting the first equation into the second we get:

$$
\begin{aligned}
B+2 &= 2B+1 \\
1 &= B
\end{aligned}
$$

Now, substituting this B value back into the first equation we get:

$$
\begin{aligned}
1+2 &= A \\
A &= 3
\end{aligned}
$$

Therefore Audrey is 3 years old and her brother is 1 year old.

3. Let n be the number of nickels, d be the number of dimes, and q be the number of quarters in my pocket.
We can make the following equations from the given information:

$$
\begin{aligned}
n+2 &= d \\
\\
n+d+q &= 10 \\
\\
0.05n+0.10d+0.25q &= 1.85
\end{aligned}
$$

Substituting the first equation into the second:

$$
\begin{aligned}
n+(n+2)+q &= 10 \\
2n+q &= 8 \\
q &= 8-2n
\end{aligned}
$$

Now substituting both this derived equation and the first equation into the third equation:

$$
\begin{aligned}
0.05n+0.10(n+2)+0.25(8-2n) &= 1.85 \\
0.05n+0.10n+0.20+2-0.50n &= 1.85 \\
-0.35n &= -0.35 \\
n &= 1
\end{aligned}
$$

Substituting these values into the first equation and the equation we derived, we get that $d = 1+2 = 3$, and $q = 8-2(1) = 6$.

Therefore there are 6 quarters, 3 dimes and 1 nickel.

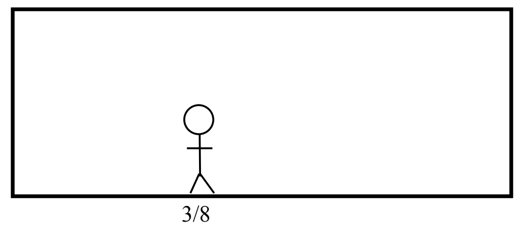
4. The first person will shake hands with 8 other people, the second person will shake hands with 7 other people (since the handshake between the first and the second person is already accounted for). This will go on for all nine people until we get the sum:

$$8+7+6+5+4+3+2+1+0=36$$

After the tenth person comes, a total of 40 handshakes were made. This means that the tenth person shook $40 = 36 = 4$ hands.

5. You know, if you were to run $\frac{3}{8}$ back, the train would just be entering the tunnel.

If you were to run that same $\frac{3}{8}$ towards the far end of the tunnel, you would be $\frac{3}{8} + \frac{3}{8} = \frac{6}{8}$ of the way through the tunnel when the train enters the tunnel.



We know that if you ran to the far end of the tunnel, you would leave at the same time the train leaves. This means for the last $\frac{8}{8} - \frac{6}{8} = \frac{2}{8} = \frac{1}{4}$ of your run, the train goes through $\frac{4}{4}$ of the tunnel.

From this, we can see that the train is going 4 times faster than you are. Since you are going 10 km/h, the train must be going $10 \times 4 = 40$ km/h.

<u>Gauss Math Competition</u>

1. Let x be the number we are looking for.

$$\begin{aligned} 3x - 12 &= 21 \\ 3x &= 33 \\ x &= 11 \end{aligned}$$

2. We know that the volume of any rectangular prism is:

$$\text{Area} = \text{Length} \times \text{Width} \times \text{Height}$$

In particular, since a cube is made of equal edge lengths:

$$\begin{aligned} \text{Area} &= \text{Length}^3 \\ 27 &= \text{L}^3 \\ \text{L} &= \sqrt[3]{27} \\ \text{L} &= 3 \end{aligned}$$

Therefore the length of each edge is 3 m.

3.

Starting Amount	**Calculations**	**Amount Left**
100%	100% × 90%	90%
90%	90% × 90%	81%
81%	81% × 90%	72.9%
72.9%	72.9% × 90%	65.61%
65.61%	65.61% × 90%	59.049%
59.049%	59.049% × 90%	53.1441%
53.1441%	53.1441% × 90%	47.28969%

We can see that after 7 uses of the bar of soap, there is less than half (50%) of the bar remaining.

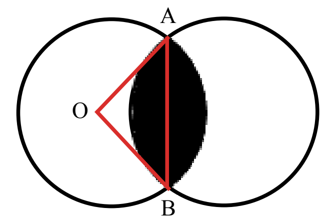
4. It can be seen that the ones digit in $3 \times P$ must be 1, as there is a 1 in the ones position of the final product. This means that $P = 7$. If this is true, we are left with:

$$\begin{array}{r} 397 \\ \underline{\times Q3} \\ 32\ 951 \end{array}$$

This can be broken up as follows:

$$\begin{aligned} 397 \times Q3 &= 32\ 951 \\ (397 \times 3) + (397 \times Q \times 10) &= 32\ 951 \\ 1\ 191 + 3\ 970Q &= 32\ 951 \\ 3\ 970Q &= 31\ 760 \\ Q &= 8 \\ \\ Q + P &= 8 + 7 \\ &= 15 \end{aligned}$$

5. Let A and B be the two intersection points of the circles.



Draw two lines from the center of the circle to these points, so that $OA = OB = 10$ cm

Constuct the line AB which splits the shaded area in half.

From the given information, $25\% = \frac{1}{4}$ of the circumference of each circle is inside of the other circle. This means that

$$\angle AOB = \tfrac{1}{4}360^\circ = 90^\circ$$

From this, it can be concluded that the area of the sector AOB is

$$AOB = \tfrac{1}{4}\pi(10)^2 = 25\pi \text{ cm }^2$$

And it can also be concluded that the area of $\triangle AOB$ is

$$\triangle AOB = \tfrac{OB \times OA}{2} = \tfrac{10 \times 10}{2} = 50 \text{ cm }^2$$

So the area of the shaded area is twice the difference between the area of the sector AOB and $\triangle AOB$

$$2 \times (25\pi \text{ cm }^2 - 50 \text{ cm}^2) \approx 57.08 \text{ cm }^2$$

Data Analysis

1.

$$\begin{aligned} \text{Mean} &= \frac{3+5+6+3+3+4+6+2+9}{9} \\ &\approx 4.56 \end{aligned}$$

2. The *range* is the difference between the largest and smallest values of a set of data.

3. Answers may vary.

4.

$$\begin{aligned} 30\% + 50\% + 10\% &= 90\% \\ (100\% - 90\%)(50) &= 5 \end{aligned}$$

5 people from the group that was polled, liked chips the best.

5. Consider first the decimal value of each time. (Letting 12:00 am be 0)

Time	**Decimal Value**
10:00 am	10
6:45 am	6.75
7:20 am	$7\ \frac{20}{60}$
9:00 am	9
8:30 am	8.5
6:00 am	6
8:10 am	$8\frac{10}{60}$

Note: Two of the times are mixed fractions

$$\begin{aligned} \text{Mean} &= \frac{10 + 6.75 + 7\frac{20}{60} + 9 + 8.5 + 6 + 8\frac{10}{60}}{7} \\ &= \frac{55.75}{7} \\ &\approx 7.964 \\ &\approx 7{:}58 \text{ am} \end{aligned}$$

Therefore the average starting time is approximately 7:58 am.

Area/Angles

1. Since ST is a straight line, the sum of all the angles must be 180°

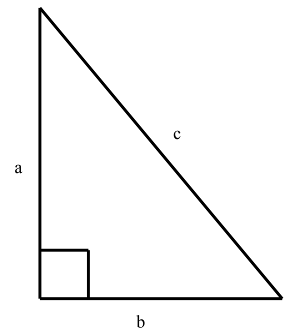
$$\begin{aligned} 5x &= 180^\circ \\ x &= 36^\circ \end{aligned}$$

2. Using the “Z” Rule, you can find the corresponding angle of 40°. After this angle is found, you can see that the triangle with this angle, and the angle we are looking for, is an isoscelesl triangle. This means

$$\begin{aligned} 40^\circ + 2x^\circ &= 180^\circ \\ 2x^\circ &= 140^\circ \\ x &= 70^\circ \end{aligned}$$

3. For any right angled triangle:

$$c^2 = a^2 + b^2$$



In any right-angled triangle, the longest side of the triangle is the hypotenuse (c)

From the question, it can be concluded that:

$$a^2 + b^2 + c^2 = 128$$

Substituting the first equation into the second:

$$\begin{aligned} c^2 + c^2 &= 128 \\ 2c^2 &= 128 \\ c^2 &= 64 \\ c &= \sqrt{64} \\ &= 8 \end{aligned}$$

Therefore the length of the longest side is 8 cm

4. Before pouring the water, the whole rectangular prism is full of water.

$$\begin{aligned} \text{Volume}_{\text{Rectangular Prism}} &= \text{Length} \times \text{Width} \times \text{Height} \\ &= 5 \text{ cm} \times 4 \text{ cm} \times 8 \text{ cm} \\ &= 160 \text{ cm}^3 \end{aligned}$$

Once the water is poured, the sum of the two volumes of water is equal to the original amount of water:

$$\begin{aligned} 160 \text{ cm}^3 &= (4 \times 5 \times h) + (\pi \times 5^2 \times h) \\ h &= \frac{160}{20 + 25\pi} \\ h &\approx 1.62 \end{aligned}$$

Therefore, the height is approximately 1.62 cm.

5. Looking at the square, it can be seen that the diagonal is also the hypotenuse of a right-angled triangle, so we can use the Pythagorean theorem to find the length of the diagonal(d):

$$\begin{aligned} 5^2 + 5^2 &= d^2 \\ d^2 &= 50 \\ d &= \sqrt{50} \end{aligned}$$

The diagonal is also the diameter of the circle. It can be seen that the difference between the area of the circle and the area of the square is the shaded area

$$\begin{aligned} \text{Area of Circle} &= (\pi)\left(\frac{\sqrt{50}}{2}\right)^2 \\ &= \pi\left(\frac{50}{4}\right) \\ &\approx 39.27 \end{aligned}$$

$$\begin{aligned} \text{Area of Square} &= 5 \times 5 \\ &= 25 \end{aligned}$$

$$39.27 - 25 \approx 14.27$$

Therefore the area of the shaded region is approximately 14.27 cm^3