Intermediate Math Circles
February 15, 2012
Contest Preparation II

Answers:

Problem Set 3:

1.	C	2.	B	3.	110°	4.	$8+8\sqrt{3}$	5.	21
6.	7	7.	E	8.	B	9.	D	10.	E

Problem Set 4:

1.	D	2.	A	3.	A	4.	C	5.	E
6.	B	7.	D	8.	E	9.	C	10.	E
11.	D	12.	D	13.	C				

Problem Set 5:

1.	B	2.	D	3.	D	4.	B	5.	C
6.	D	7.	E	8.	B	9.	E	10.	C
11.	D	12.	E						

Solutions:

Problem Set 3:

1. The answer is C since if we substitute $x = 3$ into $2(x^2 + 9)$ we get $2(3^2 + 9) = 2(9 + 9) = 36$, which is an even number.

2. The first step to solving this problem is to figure out what distance each of Jim's steps cover. Since we know that for every 3 steps that Carly takes, Jim walks 4 steps, we can write this relationship as $3x = 4y$ where x represents the distance that Carly covers per step and y represents the distance that Jim covers per step. But we know what x is since we know that each of Carly's steps are 0.5 m. So,

$$\begin{aligned} 3x &= 4y \\ 3 \times 0.5 &= 4y \\ 1.5 &= 4y \\ y &= 1.5 \div 4 \\ y &= 0.375 \end{aligned}$$

Thus each of Jim's steps covers 0.375 m. So in 24 steps Jim covers $0.375 \times 24 = 9$ m and the correct answer is B.

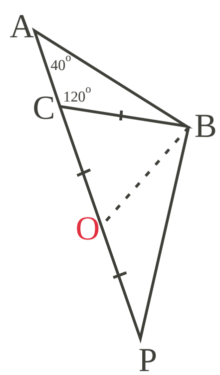
Solutions (continued):

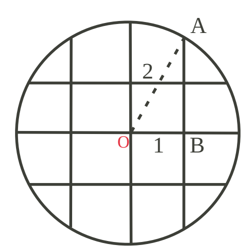
Problem Set 3:

3. Refer to diagram below left. Since $\angle ACB = 120°$ and $\angle CAB = 40°$, then $\angle CBA = 180° - 120° - 40° = 20°$. Let the midpoint of CP be O. Then CB=CO=OP and $\triangle COB$ is an isosceles triangle. $\angle OCB = 180° - 120° = 60°$. So, in isosceles triangle BCO, $\angle COB = \angle OBC = \frac{(180°-60°)}{2} = 60°$. Then $\triangle COB$ is an equilateral triangle and $OB = OC = CB$ follows. Since $CP = CB = OB$, $\triangle OPB$ is an isosceles triangle, and $\angle PCB = 120°$,

$$\begin{aligned}\angle OPB &= \angle PBO \\ &= \frac{180° - 120°}{2} \\ &= 30°\end{aligned}$$

$\therefore \angle ABP = 20° + 60° + 30° = \mathbf{110}°$.





4. Refer to diagram above right. The line containing AB is parallel to a diameter and one unit from it, $OB = 1$. Since OA is a radius, $OA = 2$. By the Pythagorean Theorem, $AB = \sqrt{2^2 + 1^2} = \sqrt{3}$. The length of the full chord containing AB is $2AB = 2\sqrt{3}$. There are 4 such chords. Adding the two diameters and the four additional chaords, the total length is: $4 \times 2 + 2\sqrt{3} \times 4 = 8 + 8\sqrt{3}$

5. Since the integer is between 200 and 300, the first digit must be 2. Then the middle digit must be at least 3 and at most 8. The chart summarizes the possibilities.

Middle Digit	Choices for Last Digit
3	$4, 5, 6, 7, 8, 9$
4	$5, 6, 7, 8, 9$
5	$6, 7, 8, 9$
6	$7, 8, 9$
7	$8, 9$
8	9

Therefore, there are a total of $6 + 5 + 4 + 3 + 2 + 1 = 21$ such integers.

6. Since 3 students do not own either a bicycle or a skateboard, that leaves 42 students that own a bicycle, a skateboard, or both. Since 27 students said they own a bicycle and 22 students said they own a skateboard, the total if the students only owned one or the other is $27 + 22 = 49$ students. But since there are only 42 students in the class that means there are $49 - 42 = 7$ students that own both a bicycle and a skateboard.

7. The area of the shaded center is $\pi(1)^2 = 1\pi$. The area of the inner white stripe is $\pi(2)^2 - \pi = 3\pi$. The area of the outer shaded stripe is $\pi(3)^2 - \pi(2)^2 = 5\pi$. The area of the largest white stripe is $\pi(4)^2 - \pi(3)^2 = 7\pi$. The total shaded area is $\pi + 5\pi = 6\pi$. The total area is $\pi(4)^2 = 16\pi$. Then the ratio of the shaded area to the total area is $6\pi : 16\pi = 3 : 8$ (E).

8. Since the area of square $ABCD = 64$, it follows that $AB = BC = CD = DA = 8$. Since F, G, H, and E are midpoints of the sides of square $ABCD$, $AF = FB = BG = GC = CH = HD = DE = EA = 4$. Using Pythagoras' Theorem, $FG^2 = FB^2 + BG^2 = 4^2 + 4^2 = 32$ and $FG = 4\sqrt{2}$ follows.

   Since $EFGH$ is a square, $EF = FG$ and the area of square $EFGH = (4\sqrt{2})^2 = 32$. Also, since $EFGH$ is a square with K, L, M, and J midpoints of the sides, $FJ = FK = LH = HM = \frac{1}{2}FG = 2\sqrt{2}$. $\triangle JFK$ and $\triangle MHL$ are right triangles since they are located in two corners of square $EFGH$. Then the area of $\triangle FJK =$ area $\triangle MHL = \frac{HM \times LH}{2} = \frac{2\sqrt{2} \times 2\sqrt{2}}{2} = 4$.

$$\begin{aligned} \text{Finally, Area } JKLM &= \text{Area } EFGH - \text{Area } \triangle FJK - \text{Area } \triangle MHL \\ &= 32 - 4 - 4 \\ &= 24 \text{ units}^2 \end{aligned}$$

   The correct answer is B.

9. Notice that $1+2+3+4+...+12+13=91$ and $1+2+3+...+12+13+14=105$ using the formula $\frac{n(n+1)}{2}$, for the sum of the positive integers 1 to n. So the 100th digit must be the number that occurs 14 times. Since we have 5 numbers that cycle and the remainder of $14 \div 5$ is 4, the 100^{th} digit must be the fourth digit in the cycle, namely 4 and the correct answer is D.

10. The diagonal on any face of the cube has length $\sqrt{2^2+2^2} = 2\sqrt{2}$ using the Pythagorean Theorem. Then half of the diagonal has length $\sqrt{2}$. Therefore, by the Pythagorean Theorem again, RQ has length $\sqrt{2^2+(\sqrt{2})^2} = \sqrt{6}$. The correct answer is E.

Problem Set 4:

1. There are 26 jelly beans in total. The probability of randomly selecting a blue jelly bean is $\frac{8}{26} = \frac{4}{13}$. The correct answer is D.

2. See past contest Cayley 1997, question 5.

3. See past contest Cayley 2008, question 7.

4. See past contest Cayley 1998, question 8.

5. See past contest Cayley 1998, question 9.

6. See past contest Cayley 1997, question 12.

7. Let x represent the side length of each edge of the cube. The formula for surface area of a cube is $SA = 6x^2$. Then $6x^2 = 54$ and $x^2 = 9$ and $x = 3, \quad x > 0$.

   Then $Volume = x^3 = 3^3 = 27$ units^3 and the answer is D.

8. Since the mean of the seven numbers 6, 14, x, 17, 9, y, 10 is 13:

$$\begin{aligned} \frac{6+14+x+17+9+y+10}{7} &= 13 \\ \frac{x+y+56}{7} &= 13 \\ x+y+56 &= 91 \\ x+y &= 35 \end{aligned}$$

   Thus, the answer is E.

9. See past contest Cayley 2004, question 12.

10. We want the least common multiple of 6 and 8. The prime factorization of 6 is 2×3 and the prime factorization of 8 is $2 \times 2 \times 2$. We can find the LCM(6,8) by taking all the prime factors of the smaller number and multiplying them by the prime factors of the larger number that are not in the prime factorization of the smaller number. So, $\text{LCM}(6, 8) = 2 \times 3 \times 2 \times 2 = 24$ and the correct answer is E.

11. The thousands digit must be a 4. The hundreds digit can take on values from 5 to 7. Then tens digit can take on values from 6 to 8. Lastly, the ones digit can take on values from 7 to 9. The results are summarized in the chart below.

Hundreds Digit	Tens Digit	Units Digit	Number of Numbers
5	6	7, 8, 9	3
5	7	8, 9	2
5	8	9	1
6	7	8, 9	2
6	8	9	1
7	8	9	1

There are a total of 10 ascending integers between 4000 and 5000. (D)

12. We start with largest y and go down to $y = 2$. If $y = 6$ and $x = 2$ then $x^y = 2^6 = 64 > 50$. So y must be less than 6. Consider $y = 5$. Then $2^5 = 32 < 50$ but $3^5 > 50$. So there is one number, namely 2^5, for $y = 5$. For $y = 4$, there is again one number, namely $2^4 = 16$. For $y = 3$, we have two numbers, $2^3 = 8$ and $3^3 = 27$. Lastly, for $y = 2$ there are five numbers, $2^2 = 4$, $3^2 = 9$, $5^2 = 25$, $6^2 = 36$, and $7^2 = 49$. Notice that $4^2 = 16$ has already been counted. There are a total of $1 + 1 + 2 + 5 = 9$ numbers and the answer is D.

13. A two digit number can be written in the form $10a + b$ where a and b are each digits from 1 to 9. Neither a nor b can be 0 since that would make one of $10a+b$ or $10b + a$ a single digit number. We want the difference between the number and its reversal to be 54.

$$\begin{aligned} 10a + b - 54 &= 10b + a \\ 9a - 9b &= 54 \\ a - b &= 6 \end{aligned}$$

We want the digits a and b to differ by 6. The choices for (a, b) are $(7, 1)$, $(8, 2)$, and $(9, 3)$. $\therefore$ there are 3 two-digit numbers with the property that the difference between the number and its reversal is 54. The correct answer is C.

Problem Set 5:

1. See past contest Cayley 1999, question 8.

2. See past contest Cayley 2004, question 10.

3. Since they bought 12 T-shirts, they we able to get 8 T-shirts at the regular price and 4 T-shirts at the sale price of \$1. Let x represent the regular price, in dollars, of one T-shirt. Then:

$$\begin{aligned} 8x + 4 \times 1 &= 120.00 \\ 8x + 4 &= 120.00 \\ 8x &= 116.00 \\ x &= \$14.50 \end{aligned}$$

   The correct answer is D.

4. See past contest Pascal 1998, question 12.

5. See past contest Pascal 2005, question 13.

6. Since 14 divided by n leaves a remainder 2 then $14 - 2 = 12$ must be divisible by n. The divisors of 12 are $2, 3, 4, 6$ and 12. From this list, $n = 2$ must be removed since 14 is divisible by 2. So there are 4 possible values of n, namely $3, 4, 6$, and 12. The correct answer is D.

7. If they cut 90% of the lawn using the riding mower, it will take 90% of the time it would take to cut the whole lawn. It will take $70 \times 0.90 = 63$ minutes to cut 90% of the lawn using the riding mower. The remaining 10% of the lawn will be cut using the push mower. It will take 0.1×5 h $= 0.1 \times 300$ minutes $= 30$ minutes to cut 10% of the lawn using the push mower. The total time to cut the lawn is $63 + 30 = 93$ minutes and the answer is E.

8. To find the number of integers between $\sqrt{40}$ and $\sqrt{400}$ we need to find the closest integer larger than $\sqrt{40}$ and the closest integer less than $\sqrt{400}$ and then count. Since $\sqrt{40} < \sqrt{49} = 7$, 7 is the smallest integer in between $\sqrt{40}$ and $\sqrt{400}$. Since $\sqrt{400} = 20$, 19 is the largest integer in between $\sqrt{40}$ and $\sqrt{400}$. There are $19 - 7 + 1 = 13$ integers in between $\sqrt{40}$ and $\sqrt{400}$. It is very easy to miscount the number of integers. The correct answer is B. (If we list the integers we can verify the count: 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19.)

9. See past contest Cayley 1997, question 17.

10. From the question, we know that d must be 5, since a number is divisible by 5 if it ends in 0 or 5 and 0 is not on the list. So the number is $abc5ef$.

    Now $def = 5ef$ is divisible by 11. A number is divisible by 11 if the sum of its odd positioned digits minus the sum of its even positioned digits is divisible by 11. In this case, $5 - e + f$ must be divisible by 11. This is only possible if $5 - e + f = 0$. The sum $5 - e + f = 11$ is not possible since e is at least 1 and f is at most 6. The only way to have $5 - e + f = 0$ is if $e = 6$ and $f = 1$. So we now have $abcdef = abc561$.

    Then we have that $cde = c56$ is divisible by 3. A number is divisible by 3 if the sum of its digits is divisible by 3. So $c + 11$ must be divisible by 3. The only digits left for c are 2, 3 and 4. Only $c = 4$ produces a sum divisible by 3, namely 15. The number is $abcdef = ab4561$.

    There are 2 choices left for a and b, namely 2 and 3. $abc = ab4$ is divisible by 4. A number is divisible by 4 if its last 2 digits are divisible by 4. This only works if $b = 2$ and $a = 3$, since 24 is divisible by 4 and 34 is not. $\therefore abcdef = 324561$ and $a = 3$. The correct answer is C.

11. $n(n+1)$ is divisible by 3 if either n is divisible by 3 or $n+1$ is divisible by 3. If n is divisible by 3, so is $n(n+1)$ and the remainder is 0. If $n+1$ is divisible by 3 then so is $n(n+1)$ and the remainder is 0. Hence the only time that $n(n+1)$ is not divisible by 3 is if neither n nor $n+1$ is divisible by 3. This means that n divided by 3 must have a remainder 1 and $n+1$ divided by 3 must have a remainder 2 (if n divided by 3 has remainder 2 then $n+1$ will be divisible by 3). Let $n = 3q + 1$ and $n + 1 = 3q + 2$, where q is some quotient. Then

$$\begin{aligned} n(n+1) &= (3q+1)(3q+2) \\ &= 9q^2 + 9q + 2 \end{aligned}$$

    The first 2 terms are both divisible by 3, leaving a remainder of 2.

    Hence the only possible remainders are 0 and 2 and the correct answer is D.

12. See past contest Cayley 2009, question 22.