Intermediate Math Circles
February 15, 2012
Contest Preparation II

The Pascal, Cayley and Fermat Contest are all written on the same day, Thursday, February 23, 2012. The deadline for registration has past but if you missed the deadline, approach the supervisor in your school or the math department head to see if there is something you can do.

Most of the problems that we will look at have been taken from past Pascal and Cayley contests.

Last week, students were given two problem sets. After some of the questions there was a reference to a past contest and a question number. For example,

11. When the product $(5^3)(7^{52})$ is expanded, the units digit is
(A) 5 (B) 3 (C) 9 (D) 7 (E) 0

Pascal 2002 #16

For the solution to this question, for example, go to our website www.cemc.uwaterloo.ca, to Mathematics and Computing Contests, to Past Contests, to PCF, to the 2002 Pascal Solutions and then look at the solution posted online.

If the problem does not have a contest reference below it, the solutions will be posted in the Math Circles section of our website found under Web Resources, Math Circles, math circles materials, Winter 2012, Intermediate and Feb-08/12.

Warm-Up #1: 2000 Pascal #22

A wooden rectangular prism has dimensions 4 by 5 by 6. This solid is painted green and then cut into 1 by 1 by 1 cubes. The ratio of the number of cubes with exactly two green faces to the number of cubes with three green faces is

(A) 9 : 2 (B) 9 : 4 (C) 6 : 1 (D) 3 : 1 (E) 5 : 2

Since the solution can be found online, it will not be reprinted here. However, as a quick check, the correct answer is (A) 9 : 2.

Warm-Up #2: 2004 Cayley #23

A soccer ball rolls at 4 m/s towards Marcos in a direct line from Michael. The ball is 15 m ahead of Michael who is chasing the ball at 9 m/s. Marcos is 30 m away from the ball and is running towards it at 8 m/s. The distance between Michael and Marcos when the ball is touched for the first time by one of them is closest to

(A) 2.00 m (B) 2.25 m (C) 2.50 m (D) 2.75 m (E) 3.00 m

Since the solution can be found online, it will not be reprinted here. However, as a quick check, the correct answer is (C) 2.50 m.

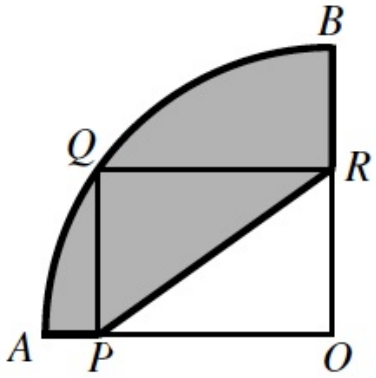
Eliminate some distractors:

1. A chocolate drink is 6% pure chocolate, by volume. If 10 litres of pure milk are added to 50 litres of this drink, the percent of chocolate in the new drink is
(A) 5 (B) 16 (C) 10 (D) 3 (E) 26

Note: Right away we can eliminate some obvious wrong answers. For instance, we can eliminate any number greater than 6 since we are adding pure milk with no percentage of chocolate to a chocolate drink that has 6% chocolate. This means that the percentage of chocolate in the drink will decrease. So, we can see that B, C and E are not correct. Now there is a 50-50 chance on getting the answer correct even if you did not know how to solve the problem.

Solution: Since 6% of the 50 L mixture is chocolate, to begin with there is $0.6 \times 50 = 3$ L of chocolate. After adding 10 L of milk there would be 3 L of chocolate in $50 + 10 = 60$ L of mixture. The new percentage is $\frac{3}{60} = 0.05 = 5\%$. The correct answer is (A).

2. In the diagram, AOB is a quarter circle of radius 10 and $PQRO$ is a rectangle of perimeter 26. The perimeter of the shaded region is



(A) $7 + 5\pi$ (B) $13 + 5\pi$ (C) $17 + 5\pi$ (D) $7 + 25\pi$ (E) $17 + 25\pi$

Note: By doing some work, we can eliminate some possibilities. Since the radius of the circle is 10, we can find the circumference to be $2\pi r = 20\pi$. So if we want to find the arc length of a quarter of this circle (because the question asks for the perimeter) then the arc length would be $\frac{20\pi}{4} = 5\pi$. So we can conclude that a number plus 25π is highly unlikely and we can eliminate both D and E. The diagonals of a rectangle are equal in length so $PR = OQ = 10$, the radius of the circle. We can probably eliminate (A). We now have a 50% chance of choosing the correct answer.

Solution: Let $OP = a$ and $OR = b$. The perimeter of the rectangle is 26 so $2a + 2b = 26$ and $a + b$ follows. Since $OA = 10$ and $OP = a$, $AP = 10 - a$. Similarly, $BR = 10 - b$. Then $AP + BR = 10 - a + 10 - b = 20 - (a + b) = 20 - 13 = 7$. The perimeter is the length of arc $AQB + AP + BR + PR = 5\pi + 7 + 10 = 5\pi + 17$. The correct answer is (C).

Try Plugging In Values:

1. The smallest positive integer x for which the sum $x + 2x + 3x + 4x + ... + 100x$ is a perfect square is
(A) 50 (B) 100 (C) 101 (D) 202 (E) 5050

Note: Try plugging in each possible answer given. Since you want the smallest positive integer, start with (A) and work your way up.

Solution:

$x+2x+3x+4x+...+100x = x(1+2+3+\cdots+100) = x\left[\frac{100\times101}{2}\right] = 5050x = 2(25)(101)x$. Both 2 and 101 are prime factors of $5050x$ and 25 is a perfect square factor of 5050. In order for $5050x$ to be a perfect square, another factor of 2 and 101 is required. Therefore, the smallest value of x is $2 \times 101 = 202$ and the correct answer is (D).

2. The products mn and rs are both 24. Both m and r have values between 1 and 24. If m is less than r, then s is

(A) less than 24 and less than n (B) greater than 24 and less than n
(C) equal to n (D) greater than 24 and greater than n
(E) less than 24 and greater than n

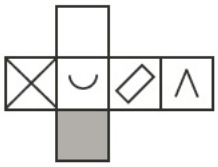
Note: Try different values for m and r where the condition holds. Then find the corresponding values of s and n and see which answer fits. The following "solution" illustrates how choosing specific numbers eliminates all but one choice.

Solution:

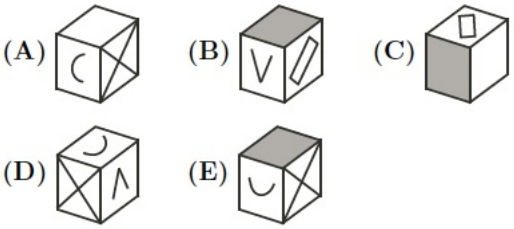
Let $m = 2$ and $r = 4$ then $n = 12$ and $s = 6$.
(C) is clearly false since $s \neq n$.
Option (B) and (D) are eliminated since $s \not> 24$.
Option (E) is eliminated since $s < n$.
The only option left is option (A).

Try constructing the object!

1. The diagram shows a piece of cardboard that can be folded to make a cube. The cardboard has designs on one side only.



Which one of the following cubes can be made from this cardboard?



Cayley 2008 #8

Note: If you are having trouble visualizing what the cube will look like from the net given, try using scrap paper to create the object and use that to figure out which answer is correct.

Since the solution can be found online, it will not be reprinted here. However, as a quick check, the correct answer is (B).

Some Final Thoughts

- Students should complete as many of the problems from Problem Set 3, 4, 5, and 6 as possible.
- Do an old contest from the type you are writing on Feb. 23.
- Choose a more recent contest.
- If you have done them all, do some more.
- Concentrate on the higher number problems.
- Learn from posted solutions.
- Enjoy problem solving.