Intermediate Math Circles
February 22, 2012
Contest Preparation II

Answers:

Problem Set 6:

1.	C	2.	A	3.	B	4.	[-6,6]	5.	$T = 3, U = 2$ and $T = 8, U = 6$
6.	69375	7.	C	8.	A	9.	C	10.	E

Australian Mathematics Competition - Intermediate 9:

1.	D	2.	E	3.	C	4.	D	5.	B
6.	D	7.	A	8.	A	9.	A	10.	D

Problem Set 6 Solutions:

1. When the sum of each pair is the same, the smallest number in the list must be grouped with the largest number in the list. The smallest number in the list is 9, the largest is 53, and $9 + 53 = 62$. Therefore, the sum of every pair is 62. Since $62 - 15 = 47$, the number paired with 15 is 47. The answer is C.

2. See past contest Pascal 2005, question 18.

3. First of all, there are 499 positive integers less than 500. Let's first count the numbers less than 500 that are divisible by 2. All even numbers under 500 are divisible by 2. There are $498 \div 2 = 249$ positive even numbers under 500. Next let's count the numbers less than 500 that are divisible by 3. The largest number smaller than 500 that is divisible by 3 is 498, where $498 \div 3 = 166$. Hence there are 166 positive integers under 500 that are divisible by 3. So far we have $499 - 249 - 166 = 85$ positive integers that are not divisible by 2 nor 3. But we've double counted the numbers that are both divisible by 2 and 3. By divisibility rules, we know that if a number is divisible by both 2 and 3, then it is divisible by 6. Therefore, let's add back the numbers that are divisible by 6. The largest number less than 500 divisible by 6 is 498, and $498 \div 6 = 83$. So, there are 83 numbers under 500 that are divisible by 6. Therefore there are a total of $499 - 249 - 166 + 83 = 167$ numbers under 500 that are not divisible by 2 or 3. The answer is B.

4. The largest possible interval for S occurs between $x - y$ is at a minimum and at a maximum. The minimum value of S occurs at $4 - 10 = -6$, the maximum value of S occurs at $12 - 6 = 6$. Hence the largest possible interval for S is $[-6, 6]$.

5. A number is divisible by 36 if it is divisible by all possible combinations of its prime factors, namely $36 = 2 \times 2 \times 3 \times 3$. Since the divisibility rules for 4 and 9 are known, we can use the fact that a number is divisible by 36 if it is divisible by both 4 and 9. A number is divisible by 4 if its last 2 digits are divisible by 4. Both 72 and 76 are divisible by 4 so we have 2 choices for U: $U = 2$ or $U = 6$.

   Case 1: Consider $U = 2$

   If $U = 2$ then the number is $9T672$. A number is divisible by 9 if the sum of its digits is divisible by 9. Then $9T672$ is divisible by 9 only when $T = 3$ since the digit sum 27 is divisible by 9. No other value of T produces a digit sum divisible by 9.

   Case 2: Consider $U = 6$

   If $U = 6$ then the number is $9T676$. The only choice for T such that $9T676$ is divisible by 9 is $T = 8$ since the digit sum 36 is the only possible sum divisible by 9. No other value of T produces a digit sum divisible by 9.T

   $\therefore$ the possible values for T and U are: $U = 2$, $T = 3$, and $U = 6$, $T = 8$

6. Let's give a name to these three-digit numbers for easy description, we'll call them "completely-odd". There are 5 odd one-digit numbers (namely 1,3,5,7, and 9). Hence there are 5 choices for the hundred's digit, 5 choices for the ten's digit, and 5 choices for the one's digit. We have a total of $5 \times 5 \times 5 = 125$ completely-odd numbers. Next, we must realize that adding three-digit numbers together is the same as adding the hundreds digits, multiplying by 100, adding the tens digits, multiplying by 10, and adding the ones digits, multiplying by 1 (For example, $531 + 317 = 100 \times (5 + 3) + 10 \times (3 + 1) + 1 \times (1 + 7) = 848$). The 1 in the hundred's digit occurs 25 times (since there are 25 out of the completely-odd numbers with a 1 in the hundred's digit), the 1 in the ten's digit also occurs 25 times, and the 1 in the one's digit also occurs 25. The same goes for the digits 3, 5, 7,

and 9. Hence the sum of all completely-odd numbers is:

$$\begin{aligned}
& 25 \times [100 \times (1+3+5+7+9) + 10 \times (1+3+5+7+9) \\
& +1 \times (1+3+5+7+9)] \\
= & [25 \times (1+3+5+7+9)] \times (100+10+1) \\
= & 625 \times 111 \\
= & 69375
\end{aligned}$$

7. See past contest Pascal 2005, question 20.

8. All even numbers are divisible by 2. No odd numbers are divisible by 2. We will expand 90! and try to divide it by 2 as many times as possible until the quotient is odd, then we know that we can no longer divide by 2. (Note, a number n divided by, say $8 = 2^3$, is the same as $n \div 2 \div 2 \div 2$). From 1 to 90, there are 45 even numbers. So all of these are divisible by 2. Then we can divide 90! by 2 at least 45 times. Performing this operation (dividing by 2 forty-five times) leaves a quotient of (45)(89)(44)(87)...(2)(3)(1)(1). Then we notice that every 4 of these numbers are still even. Namely, from 1 to 45, we have $44 \div 2 = 22$ even numbers left. So we can divide 2 into this quotient another 22 times. We get a new quotient of (45)(89)(22)(87)(43)(85)(21)...(1)(3)(1)(1). Now every 8 numbers is a even number. Namely from 1 to 22, there are $22 \div 2 = 11$ even numbers. So we can divide 2 into this new quotient another 11 times. Carrying on this pattern we get that there are $10 \div 2 = 5$ more even numbers that occur every 16 numbers; then there are $4 \div 2 = 2$ more even numbers that occur every 32 numbers, and finally $2 \div 2 = 1$ more even number that occur every 64 numbers.

   In total, the number of factors of 2 in 90! is $45 + 22 + 11 + 5 + 2 + 1 = 86$ (i.e. 2^{86} divides 90!). Therefore, the highest exponent n such that 2^n divides evenly into 90! is 86. The answer is A.

9. See past contest Cayley 1997, question 22.

10. See past contest Cayley 1998, question 22.

Australian Mathematics Competition - Intermediate 9 Solutions:

1.

$$\begin{aligned}(7a+5b)-(5a-7b) &= 7a+5b-5a+7b \\ &= 2a+12b \qquad \text{(D)}\end{aligned}$$

2. The sum of angles along a straight line is 180°. So,

$$\begin{aligned}60^\circ+20^\circ+54^\circ+x^\circ+x^\circ &= 180^\circ \\ 134+2x &= 180 \\ 2x &= 46 \\ x &= 23 \qquad \text{(E)}\end{aligned}$$

3.

$$\begin{aligned}\frac{\sqrt{20+x^2}}{\sqrt{20-x^2}} &= \frac{\sqrt{20+4^2}}{\sqrt{20-4^2}} \\ &= \frac{\sqrt{20+16}}{\sqrt{20-16}} \\ &= \frac{\sqrt{36}}{\sqrt{4}} \\ &= \frac{6}{2} \\ &= 3 \qquad \text{(C)}\end{aligned}$$

4. Dividing a number n by, say $8 = 2^3$, is the same as $n \div 2 \div 2 \div 2$. How many times can we divide 2 into 1000000?

We have:

$$\begin{aligned}1000000 \div 2 &= 500000 && (1)\\ 500000 \div 2 &= 250000 && (2)\\ 250000 \div 2 &= 125000 && (3)\\ 125000 \div 2 &= 62500 && (4)\\ 62500 \div 2 &= 31250 && (5)\\ 31250 \div 2 &= 15625 \quad \text{which is an odd number} && (6)\end{aligned}$$

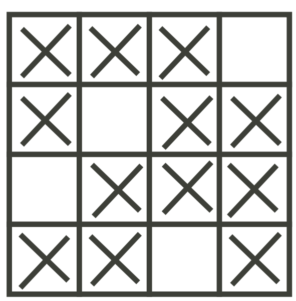
We've divided 2 into 1000000 six times. That is, 2^6 divides 1000000. The answer is D.

5. 1L of orange fruit juice is equivalent to 1000 mL of orange fruit juice. Since the 1000 mL contains 10% orange juice, there are $1000 \times 0.10 = 100$ mL of orange juice.
Let the amount of orange juice to be added be x. Now the mixture contains $(100 + x)$ mL of orange juice and $(1000 + x)$ mL of mixture. We want the amount of orange juice to be 50% of the mixture. Therefore,

$$\begin{aligned}\frac{100+x}{1000+x} &= 0.5\\ 100+x &= 0.5 \times (1000+x)\\ 100+x &= 500+0.5x\\ 0.5x &= 400\\ x &= 800\end{aligned}$$

The amount of orange juice to be added is 800 mL. The answer is B.

6. One way to approach this problem is by using little strips of paper to represent the rows. We know that there are 4 ways to arrange 3 counters in four spots. If you put each of the ways on a strip of paper, you can rearrange the strips to satisfy the conditions. By doing this, we can see that you can arrange 12 counters in the boxes and satisfy all the conditions required. There are multiple solutions to this problem, but one of them is shown in the diagram below. It is not too difficult to show that 13 is not possible and that more than 10 is possible. You can reduce the possible answers to 11 and 12 quite easily.The answer is D.



7. All positive integers less than 1000 have either 3 digits, 2 digits or 1 digit. The only 1 digit number whose digit sum is 6 is 6 itself.

   For two-digit numbers, there are 6 choices for the ten's digit, namely, {6, 5, 4, 3, 2, 1}. Each of these choices for the ten's digit has a unique corresponding one's digit. Therefore, there are 6 two-digit numbers whose digit sum is 6.

   For three-digit numbers, there are 6 choices for the hundred's digit. With each of these choices of the hundred's digit, the remaining 2 digits' sum needs to add up to 6 minus the value of hundred's digit. This gives us that the ten's and one's digit must add up to 0, 1, 2, 3, 4, or 5, depending on the value of the hundred's digit. Then, by the two-digit number argument, there are respectively, 1, 2, 3, 4, 5, or 6 choices for the ten's digit, and each of these choice of the ten's digit correspond to a unique one's digit. Therefore, there are a total of $1 + 2 + 3 + 4 + 5 + 6 = 21$ such three-digit numbers.

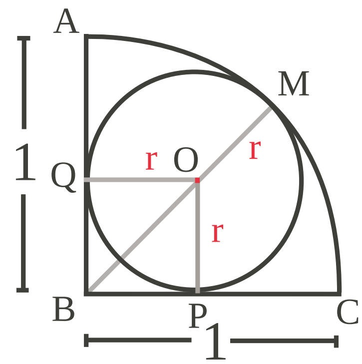
   Hence the number of positive integers less than 1000 whose sum of digits equals 6 is: $21 + 6 + 1 = 28$. The answer is A.

8. Consider the ratio of John's money compared with Kevin and Robert's combined money. Before gambling, their ratio is $7 : 11$. After gambling, their ratio is $6 : 9$. Since $6 : 9 > 7 : 11$, John must have won the \$12. The winnings came from the combined total of Kevin and Robert. Let $7x$ be the amount of money John had to start and $11x$ be the combined total of Kevin and Robert at the start. Then,

$$\begin{aligned}\frac{7x+12}{11x-12} &= \frac{6}{9}\\ 66x-72 &= 63x+108\\ 3x &= 180\\ x &= 60\end{aligned}$$

   So John started with $7x = 7(60) =$\$420 dollars. The answer is A.

9. The largest circle that can be drawn in this quarter circle must be an inscribed circle that touches the quarter circle on the arc and the 2 radii. Consider the diagram below.



From the diagram, we see that $OQ \perp AB$ and $OP \perp BC$ (line from center of circle to a tangent always intersect the tangent at 90°). Then $\triangle OBP$ and $\triangle OBQ$ are two identical right-angled triangles, with $OP = OQ = BP = BQ = r$. Then OB has length, by the Pythagorean Theorem, $\sqrt{r^2 + r^2} = \sqrt{2}r$. Also note that BM is also a radius of the quarter circle. So $BM = BA = BC = 1$. But $BM = OB + OM = \sqrt{2}r + r$. Therefore,

$$\begin{aligned}
\sqrt{2}r + r &= 1 \\
r(\sqrt{2} + 1) &= 1 \\
r &= \frac{1}{\sqrt{2} + 1} \\
&= \frac{1}{\sqrt{2} + 1} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1} \\
&= \sqrt{2} - 1
\end{aligned}$$

The answer is A.

10. We'll convert $\frac{7}{10}$ and $\frac{11}{15}$ into fractions with the same denominator. Namely,

$$\frac{21}{30} < \frac{p}{q} < \frac{22}{30}$$

Notice that if $q = 30$, there is not an integer solution for p that satisfies the relationship. Let us multiply the fractions again by $\frac{2}{2}$, to get:

$$\frac{42}{60} < \frac{p}{q} < \frac{44}{60}$$

Notice that if $q = 60$, then the only integer solution for p is $p = 43$, which satisfies the relationship. The answer is B.