Intermediate Math Circles
February 29, 2012
Problem Set: Linear Diophantine Equations I
Solutions

1. a) Find an integer solution to the Diophantine equation $4389x + 2919y = 21$.

Solution
There is a solution only if gcd(4389, 2919) divides 21.

$$4389 = 1 \cdot 2919 + 1470 \quad (1)$$
$$2919 = 1 \cdot 1470 + 1449 \quad (2)$$
$$1470 = 1 \cdot 1449 + 21 \quad (3)$$
$$1449 = 69 \cdot 21 + 0 \quad (4)$$

So gcd(4389, 2919) = 21

$$\begin{aligned}
21 &= \underline{1470} - 1 \cdot \underline{1449} && \text{from (3)} \\
&= \underline{1470} - 1 \cdot (\underline{2919} - 1 \cdot \underline{1470}) && \text{from (2)} \\
&= 2 \cdot \underline{1470} - 1 \cdot \underline{2919} \\
&= 2 \cdot (\underline{4389} - 1 \cdot \underline{2919}) - 1 \cdot \underline{2919} && \text{from (1)} \\
&= 2 \cdot \underline{4389} - 3 \cdot \underline{2919}
\end{aligned}$$

So one solution is $x = 2$, $y = -3$. □

b) Find an integer solution to the Diophantine equation $4389x + 2919y = 231$.

Solution
We know that gcd(4389, 2919) = 21, so this linear Diophantine equation has a solution only if 21 divides 231.
$231 = 21 \times 11$, so we have a solution!

First: Find a solution to $4389x + 2919y = 21$. We've done this: $x = 2$, $y = -3$.
Second: Multiply by 11. So the new solution is $x = 22$, $y = -33$.
Check: $4389(22) + 2919(-33) = 231$. □

2. Find an integer solution to the Diophantine equation $212x - 37y = 1$.

Solution
There is a solution only if gcd(212, 37) divides 1.

$$\begin{aligned}
212 &= 5 \cdot 37 + 27 && (1)\\
37 &= 1 \cdot 27 + 10 && (2)\\
27 &= 2 \cdot 10 + 7 && (3)\\
10 &= 1 \cdot 7 + 3 && (4)\\
7 &= 2 \cdot 3 + 1 && (5)\\
3 &= 3 \cdot 1 + 0 && (6)
\end{aligned}$$

So gcd(212, 37) = 1

$$\begin{aligned}
1 &= \underline{7} - 2 \cdot \underline{3} \qquad \text{from (5)}\\
&= \underline{7} - 2 \cdot (\underline{10} - 1 \cdot \underline{7}) \qquad \text{from (4)}\\
&= 3 \cdot \underline{7} - 2 \cdot \underline{10}\\
&= 3 \cdot (\underline{27} - 2 \cdot \underline{10}) - 2 \cdot \underline{10} \qquad \text{from (3)}\\
&= 3 \cdot \underline{27} - 8 \cdot \underline{10}\\
&= 3 \cdot \underline{27} - 8 \cdot (\underline{37} - 1 \cdot \underline{27}) \qquad \text{from (2)}\\
&= 11 \cdot \underline{27} - 8 \cdot \underline{37}\\
&= 11 \cdot (\underline{212} - 5 \cdot \underline{37}) - 8 \cdot \underline{37} \qquad \text{from (1)}\\
&= 11 \cdot \underline{212} - 63 \cdot \underline{37}
\end{aligned}$$

So one solution is $x = 11$, $y = 63$. $\square$

3. Find an integer solution to the Diophantine equation $12x + 57y = 423$.

Solution
There is a solution only if gcd(12, 57) divides 423.

$$57 = 4 \cdot 12 + 9 \qquad (1)$$
$$12 = 1 \cdot 9 + 3 \qquad (2)$$
$$9 = 3 \cdot 3 + 0 \qquad (3)$$

So gcd(12, 57) = 3. Since 3 divides 423, there is a solution to the linear Diophantine equation.
First, find a solution to $12x + 57y = 3$:

$$\begin{aligned} 3 &= \underline{12} - 1 \cdot \underline{9} && \text{from (2)} \\ &= \underline{12} - 1 \cdot (\underline{57} - 4 \cdot \underline{12}) && \text{from (1)} \\ &= 5 \cdot \underline{12} - 1 \cdot \underline{57} \end{aligned}$$

So $12(5) + 57(-1) = 3$.
Multiply both sides by $\frac{423}{3} = 141$:

$$\begin{aligned} 141(12(5) + 57(-1)) &= 141 \times 3 \\ 12(141 \cdot 5) + 57(141 \cdot (-1)) &= 423 \\ 12(705) + 57(-141) &= 423 \end{aligned}$$

So one solution is $x = 705$, $y = -141$. □

4. Can 1000 be expressed as the sum of two integers, one of which is divisible by 11 and the other by 17? If so, determine one such way.

   Solution
   In other words, is there a solution to the linear Diophantine equation $11x + 17y = 1000$? There is a solution only if gcd(11, 17) divides 1000.

$$\begin{aligned} 17 &= 1 \cdot 11 + 6 && (1) \\ 11 &= 1 \cdot 6 + 5 && (2) \\ 6 &= 1 \cdot 5 + 1 && (3) \\ 5 &= 5 \cdot 1 + 0 && (4) \end{aligned}$$

   So gcd(11, 17) = 1. Since 1 divides 1000, there is a solution to the linear Diophantine equation.
   First, find a solution to $11x + 17y = 1$:

$$\begin{aligned} 1 &= \underline{6} - 1 \cdot \underline{5} \qquad \text{from (3)} \\ &= \underline{6} - 1 \cdot (\underline{11} - 1 \cdot \underline{6}) \qquad \text{from (2)} \\ &= 2 \cdot \underline{6} - 1 \cdot \underline{11} \\ &= 2 \cdot (\underline{17} - 1 \cdot \underline{11}) - 1 \cdot \underline{11} \qquad \text{from (1)} \\ &= 2 \cdot \underline{17} - 3 \cdot \underline{11} \end{aligned}$$

   So $11(-3) + 17(2) = 1$.
   Multiply both sides by $\frac{1000}{1} = 1000$:

$$\begin{aligned} 1000(11(-3) + 17(2)) &= 1000 \times 1 \\ 11(-3 \cdot 1000) + 17(2 \cdot 1000) &= 1000 \\ 11(-3000) + 17(2000) &= 1000 \end{aligned}$$

   So one solution is $x = -3000$, $y = 2000$.
   Therefore, 1000 can be expressed as the sum of -33000 (which is divisible by 11) and 34000 (which is divisible by 17). □

5. Can 1000 be expressed as the sum of two integers, one of which is divisible by 9 and the other by 12? If so, determine one such way.

   Solution
   In other words, is there a solution to the linear Diophantine equation $9x + 12y = 1000$? There is a solution only if gcd(9, 12) divides 1000.
   By inspection, gcd(9, 12) = 3 and since 3 does not divide 1000, there is no solution. □

6. Here's a little puzzle: start with the number 0, and at every step, you may add or subtract either the number 5 or the number 17 (that's four possible moves in total). Is it possible to eventually get to the number 1? If so, describe how.

$$0 \xrightarrow{+17} 17 \xrightarrow{-5} 12 \longrightarrow \dots$$

<u>Solution</u>
In other words, is there a solution to the linear Diophantine equation $5x + 17y = 1$?
There is a solution only if gcd(5, 17) divides 1.

$$17 = 3 \cdot 5 + 2 \tag{1}$$
$$5 = 2 \cdot 2 + 1 \tag{2}$$
$$2 = 2 \cdot 1 + 0 \tag{3}$$

So gcd(5, 17) = 1 and there is a solution to the linear Diophantine equation.

$$\begin{aligned} 1 &= \underline{5} - 2 \cdot \underline{2} \qquad \text{from (2)} \\ &= \underline{5} - 2 \cdot (\underline{17} - 3 \cdot \underline{5}) \qquad \text{from (1)} \\ &= 7 \cdot \underline{5} - 2 \cdot \underline{17} \end{aligned}$$

So $5(7) + 17(-2) = 1$.
Therefore, yes, it is possible to eventually get to the number 1. One way is to add 5 seven times, then subtract 17 twice. □