Intermediate Math Circles
March 7, 2012
Linear Diophantine Equations II

Last week: How to find **one** solution to a linear Diophantine equation
This week: How to find **all** solutions to a linear Diophantine equation

Recall: Linear Diophantine Equations are equations of the form $ax + by = c$,
where a, b, c are given integers and we are solving for integers x and y.

For example:

$$25x + 10y = 155$$
or
$$1053x + 481y = 13$$

Main question: If a, b, and c are integers, when does a solution exist and how can you find a solution to

$$ax + by = c$$

where x and y are integers?

Sometimes trial-and-error works, but this is not always very efficient.
And sometimes a solution does not even exist!

Last week we saw:

Important Fact: $ax + by = c$ has a solution if and only if $\gcd(a, b)$ divides c.

- An integer d **"divides"** another integer e if and only if there is some integer q such that $e = qb$.
- The **"greatest common divisor"** of a and b is the largest integer that divides a <u>and</u> divides b. We write it as $\gcd(a, b)$.

What's the best way to calculate $\gcd(a, b)$?

For small numbers a and b it's not too hard to find gcd by factoring a and b, but not as easy when trying to calculate

$$\gcd(104723,\ 103093) \text{ or } \gcd(3551,\ 4399)$$

You can try to find all divisors, and see which ones divide both, but that could take all day!

Instead, we'll use the **Euclidean Algorithm** and
the **Important Fact**: $\boxed{\text{If } a = qb + r, \text{ then } \gcd(a, b) = \gcd(b, r).}$

"**The Euclidean Algorithm**" for calculating $\gcd(a,\ b)$:

> Step 1: Arrange a and b so that $a \geq b$.
>
> Step 2: Write $a = qb + r$, with $0 \leq r < b$.
>
> Step 3: If $r = 0$, then b divides a, so $\gcd(a,\ b) = b$. STOP!
> If not then $\gcd(a,\ b) = \gcd(b,\ r)$.
>
> Step 4: Go to Step 2 to calculate $\gcd(b,\ r)$.

Since the numbers get smaller after each iteration, you will eventually get an answer.

Example 1: Calculate $\gcd(2173,\ 2491)$.

Solution:

$\gcd(2173,\ 2491) = \gcd(2491,\ 2173)$

$$\begin{aligned}
2491 = 1 \cdot 2173 + 318, \text{ so} \quad & \gcd(2491,\ 2173) = \gcd(2173,\ 318) \\
2173 = 6 \cdot 318 + 265, \text{ so} \quad & \gcd(2173,\ 318) = \gcd(318,\ 265) \\
318 = 1 \cdot 265 + 53, \text{ so} \quad & \gcd(318,\ 265) = \gcd(265,\ 53) \\
265 = 5 \cdot 53 + 0, \text{ so} \quad & \gcd(265,\ 53) = 53 \\
\text{We have} \quad & \gcd(2173, 2491) = 53.
\end{aligned}$$

$\square$

An application of the Euclidean Algorithm: Solving linear Diophantine equations.

In general, how do we solve $ax + by = c$ for integers x and y?

1. Calculate $\gcd(a, b)$ by using the Euclidean Algorithm.
2. Does $\gcd(a, b)$ divide c?
   a) If NO, then there is no solution to the linear Diophantine equation.
   b) If YES then
      (i) Solve $ax + by = \gcd(a, b)$ by working backwards through your steps in the Euclidean Algorithm.
      (ii) Multiply the x and y in the solution by $\dfrac{c}{\gcd(a, b)}$.

Example 2: Find integers x and y such that $2173x + 2491y = 53$.

Solution:

From the Euclidean Algorithm:

$$\begin{aligned} 2491 &= 1 \cdot 2173 + 318 \qquad (1)\\ 2173 &= 6 \cdot 318 + 265 \qquad (2)\\ 318 &= 1 \cdot 265 + 53 \qquad (3)\\ 265 &= 5 \cdot 53 + 0 \qquad (4) \end{aligned}$$

So gcd(2173, 2491) = 53, which divides $c = 53$, so there is a solution to this linear Diophantine equation.

We find the solution by working backwards through our steps from the Euclidean Algorithm:

$$\begin{aligned} 53 &= 318 - 1 \cdot 265 \quad \text{from (3)}\\ &= 318 - 1(2173 - 6 \cdot 318) \quad \text{from (2)}\\ &= 7 \cdot 318 - 1 \cdot 2173\\ &= 7(2491 - 1 \cdot 2173) - 1 \cdot 2173 \quad \text{from (1)}\\ &= 7(2491) - 8(2173) \end{aligned}$$

Therefore, a solution is $x = -8, y = 7$. (Check!)

□

Example 3: Find integers x and y such that $2173x + 2491y = 159$.

Solution:

Since gcd(2173, 2491) = 53 and 53 divides 159, we know that there is a solution to this linear Diophantine equation.

Our first step is to solve $2173x + 2491y = \gcd(2173, 2491) = 53$.

We already know that	$2173(-8) + 2491(7) = 53$
Multiplying both sides by 3:	$3(2173(-8) + 2491(7)) = 3(53)$
And so	$2173(3(-8)) + 2491(3(7)) = 159$
Therefore,	$2173(-24) + 2491(21) = 159$

So one solution is $x = -24,\ \ y = 21$.

Example 4: Find an integer solution to $2173x + 2491y = 210$.

Solution:

In Example 2 we saw that gcd(2173, 2491) = 53, and since 53 does not divide 210, there is no integer solution.

Exercise Set 1

1. Use the Euclidean Algorithm to calculate $\gcd(3689, 4182)$.

2. Find an integer solution to $3689x + 4182y = 17$, or explain why one doesn't exist.

3. Find an integer solution to $3689x + 4182y = 102$, or explain why one doesn't exist.

4. Find an integer solution to $3689x + 4182y = 100$, or explain why one doesn't exist.

Answers to Exercise Set 1:

1. $\gcd(3689, 4182) = 17$
2. $(x, y) = (-17, 15)$ is one (of many) solutions.
3. $(x, y) = (-102, 90)$ is one (of many) solutions.
4. There is no solution since $\gcd(3689,\ 4182) = 17$, and 17 does not divide 100.

So we now know that

$$ax + by = c$$

has a solution (in the integers) if and only if $\gcd(a, b)$ divides c, and we know how to find a solution if there is one.

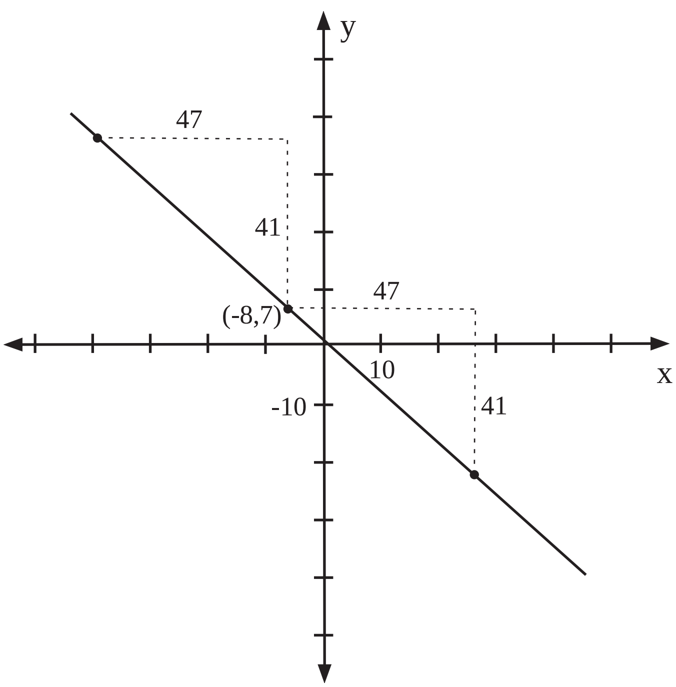
How can we find more solutions?

Let's go back to $2173x + 2491y = 53$.

In Example 2, we saw that one solution to this linear Diophantine equation is $x = -8, y = 7$.

What is happening geometrically?

This is the equation of a line through the point $(-8, 7)$:



Slope of the line: $-\dfrac{2173}{2491}$ $\left(-\dfrac{a}{b}\right)$

Can we reduce? $-\dfrac{2173}{2491} = -\dfrac{41}{47}$ $\left(-\dfrac{\frac{a}{\gcd(a,b)}}{\frac{b}{\gcd(a,b)}}\right)$

How do we find another lattice point?
left 47, up 41 $(-55, 48)$
another? right 47, down 41 $(39, -34)$
another? left 94, up 82 $(-102, 89)$
and so on...

In general:

Suppose x_0, y_0 is one solution to $ax + by = c$ and let $d = \gcd(a, b)$.
Then the **full set** of integer solutions is given by

$$x = x_0 + n\left(\frac{b}{d}\right) \text{ and } y = y_0 - n\left(\frac{a}{d}\right), \text{ where } n \text{ is any integer.}$$

So for $2173x + 2491y = 53$, since one solution is $x = -8, y = 7$, the complete solution is

$$x = -8 + \frac{2491}{53}n = -8 + 47n$$
$$y = 7 - \frac{2173}{53}n = 7 - 41n$$

where n is any integer

What if we wanted, for example, $x \geq 150$?
Then we would need $x = -8 + 47n \geq 150$
And so, $47n \geq 158$
Thus, $n \geq 3.36...$
and since n is an integer, this implies that we must have $n \geq 4$.

What if we wanted, for example, $x \leq y$?
Then we would need $x = -8 + 47n \leq 7 - 41n = y$
And so, $88n \leq 15$
Thus, $n \leq 0.17...$
and since n is an integer, this implies that we must have $n \leq 0$.

In general, to solve $ax + by = c$ for integers x and y:

1. Calculate $\gcd(a, b)$ by using the Euclidean Algorithm.
2. Does $\gcd(a, b)$ divide c?
   a) If NO, then there is no solution to the linear Diophantine equation.
   b) If YES then
      (i) Solve $ax + by = \gcd(a, b)$ by working backwards through your steps in the Euclidean Algorithm. (Or, alternatively, find a solution by inspection if this is easy to do for the particular question).
      (ii) Multiply the x and y in the solution by $\frac{c}{\gcd(a, b)}$.
      (iii) Use the formula to find all solutions to the Diophantine equation: $x = x_0 + n\left(\frac{b}{d}\right)$ and $y = y_0 - n\left(\frac{a}{d}\right)$
      (iv) Use any restrictions in the problem (for example, x and y must be non negative) to restrict n and find the solutions you are interested in.

Example 5: A trucking company has to move 844 refrigerators. They have two types of trucks, one that carries 28 refrigerators and one that carries 34 refrigerators. The company only sends out full trucks, and the trucks return empty. List all possible ways to move all the refrigerators.

Solution:

Let x be the number of "small" trucks and y be the number of "large" trucks.

Since a small truck can carry 28 refrigerators and a large truck can carry 34 refrigerators, we are interested in solving the linear Diophantine equation

$$28x + 34y = 844$$

First, let's use the Euclidean Algorithm to calculate gcd(28,34):

$$34 = 1 \cdot 28 + 6 \quad (1)$$
$$28 = 4 \cdot 6 + 4 \quad (2)$$
$$6 = 1 \cdot 4 + 2 \quad (3)$$
$$4 = 2 \cdot 2 + 0 \quad (4)$$

So gcd(28, 34) $= 2$, which divides $c = 844$, so there is a solution to this linear Diophantine equation.

We find a solution to $28x + 34y = 2$ by working backwards through our steps from the Euclidean Algorithm:

$$\begin{aligned} 2 &= 6 - 1 \cdot 4 \text{ from (3)} \\ &= 6 - 1(28 - 4 \cdot 6) \text{ from (2)} \\ &= 5 \cdot 6 - 1 \cdot 28 \\ &= 5(34 - 1 \cdot 28) - 1 \cdot 28 \text{ from (1)} \\ &= 5(34) - 6(28) \end{aligned}$$

Therefore, $28(-6) + 34(5) = 2$.
Let's use this to find a solution to $28x + 34y = 844$.

We already know that $28(-6) + 34(5) = 2$

Multiplying both sides by 422: $422(28(-6) + 34(5)) = 422(2)$

And so $28(422 \times -6) + 34(422 \times 5) = 844$

Therefore, $28(-2532) + 34(2110) = 844$

So a solution is $x = -2532$ and $y = 2110$.
Are there more solutions?

Since $x = -2532, y = 2110$ is a particular solution, the complete solution is:
$x = -2532 + n(\frac{34}{2}) = -2532 + 17n$ and $y = 2110 - n(\frac{28}{2}) = 2110 - 14n$

Can x and y be negative?

Since x and y represent trucks, we need $x \geq 0$ and $y \geq 0$:

$x \geq 0$ means that $-2532 + 17n \geq 0$, thus $17n \geq 2532$ and $n \geq 148.94...$
Since n is an integer, we need $n \geq 149$.

$y \geq 0$ means that $2110 - 14n \geq 0$, thus $14n \leq 2110$ and $n \leq 150.71...$
Since n is an integer, we need $n \leq 150$.
So $149 \leq n \leq 150$.

If $n = 149$, we obtain the solution
$x = -2532 + 149(17) = 1$ and $y = 2110 - 149(14) = 24$

If $n = 150$, we obtain the solution
$x = -2532 + 150(17) = 18$ and $y = 2110 - 150(14) = 10$

Therefore, there are two ways to move all the refrigerators. The trucking company could either send 1 small truck and 24 large trucks, or send 18 small trucks and 10 large trucks.

Exercise Set 2

1. Find **all** integer solutions to $3689x + 4182y = 17$.
   Hint: See Exercise Set 1.

2. Find **all** integer solutions to $3689x + 4182y = 102$.
   Hint: See Exercise Set 1.

3. a) Find all solutions to $12x + 57y = 3$.
   b) Find all solutions to $12x + 57y = 423$.
   c) Find all solutions to $12x + 57y = 423$ where $x \geq 0$ and $y \geq 0$.

Answers to Exercise Set 2:

1. $x = -17 + 246n,\ y = 15 - 217n$, where n is any integer.

2. $x = -102 + 246n,\ y = 90 - 217n$, where n is any integer.

3. a) $x = 5 + 19n,\ y = -1 - 4n$, where n is any integer.
   b) $x = 705 + 19n,\ y = -141 - 4n$, where n is any integer.
   c) $(x, y) = (2, 7)$ or $(x, y) = (21, 3)$