Intermediate Math Circles
March 21, 2012
Probability

This week and next week we'll be talking about probability and random events. We'll start with some warm-ups, and discuss how there is a unique limiting shape if you repeat the same event enough times. We'll work on some problems about *conditional probability*: how likely is some event to happen *given that* some other event happened? Next week, we'll discuss expected values; along the way we'll see random walks, game theory, and gambling problems.
You can contact Dave with questions at `dagpritc@uwaterloo.ca`. Solutions will go on the website.

Warm-up

If you roll a die, what is the probability that the number (on top) is a prime number?

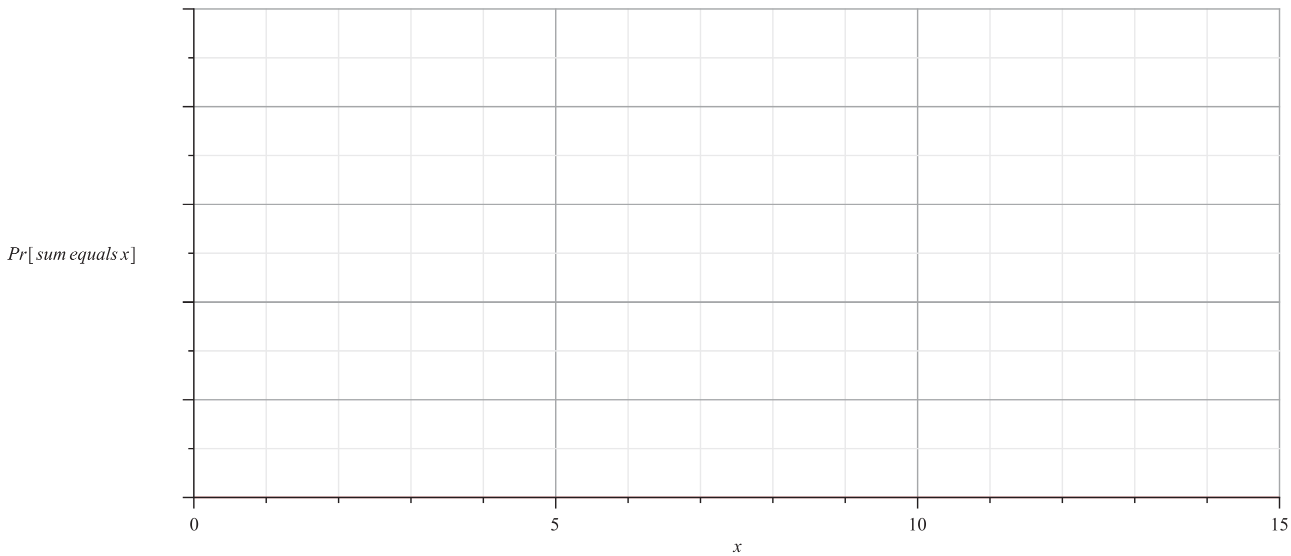
What is the general rule for computing probabilities? What is the minimum and maximum possible probability?

If you flip two coins, what is the probability that one of them comes up heads and one of them is tails?

If two events A and B are completely unrelated to each other (e.g., A is "the penny comes up heads," B is "the nickel comes up heads") what is the relation between the probability that A happens (written $\mathsf{Pr}[A]$), the probability that B happens, and the probability that both of them happen (written $\mathsf{Pr}[A \text{ and } B]$)?

We roll two dice and add up the numbers that show on top. What is the probability that the sum is 6?

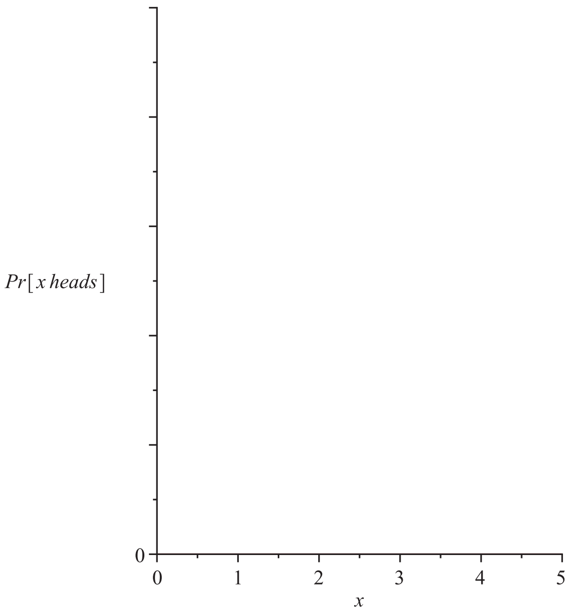
Draw a diagram showing how likely each sum is.

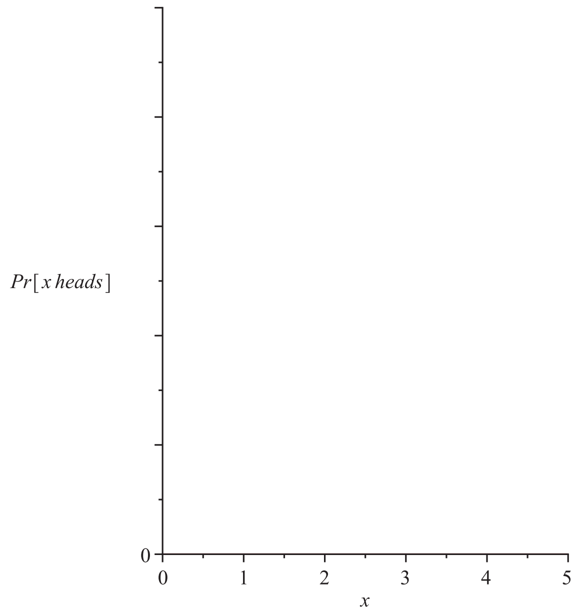


What is the probability that the sum is even? (Hint: there is a long solution but also a short solution.)

Aside: The Normal Distribution

The previous diagram was pretty stylish. What does it look like with 3 dice, 10 dice, or 100 dice? We'll check it out on the computer.
What about coins? Coins are pretty flat compared to dice so maybe a very different shape would emerge. Let's take 5 coins as an example. We'll fill in the left diagram using data collected in class. Then we'll compute the exact probabilities and fill in the diagram on the right.



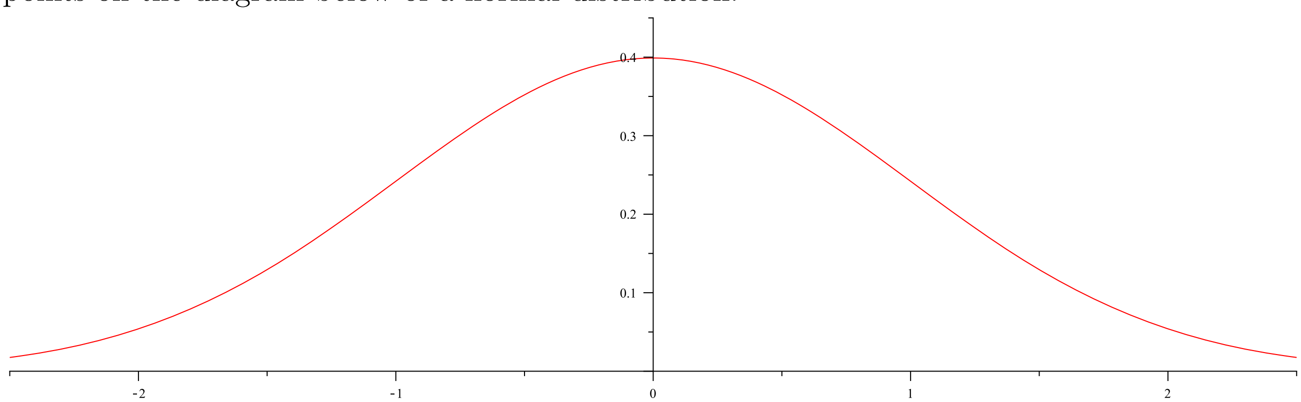


We need to recall: the number of ways to select k out of n items, written $\binom{n}{k}$ and called "n choose k," equals

$$\frac{n}{k}\frac{n-1}{k-1}\cdots\frac{n-k+1}{1}=\frac{n!}{(n-k)!k!}.$$

On the computer we'll try to experiment again with 100 coins instead of 100 dice. The shape is pretty similar, except that it has been moved and squished a little bit. What about if we use a biased coin, or an unfair die?

Next week we'll talk some more about the precise meaning of the mean and standard deviation. For *normal distributions* there is a shortcut: the mean is where the maximum/center point is located, and one standard distribution away is where the *inflection points* lie. An inflection point has maximum slope: it is where biking uphill would be hardest. Draw the mean and inflection points on the diagram below of a normal distribution.



Law and Politics, Part I

The people in Probabilia are all either blue and green. The population is 90% green and 10% blue. There was a robbery last night, and Officer Ophelia saw the robber running away. She thinks that the robber was blue. However, at night time she has a 20% chance of making a mistake in trying to figure out the colour of anybody she sees.
Bob, a blue person found near the scene of the crime, is being held in custody, and Ophelia argues that the robber was 80% likely to be blue. Bob's lawyer argues that the robber was more likely to be blue than green, so Bob should be released! Why?

Conditional Probability

Conditional probability means "the probability that something happens, **if** something else happens." This helps to keep track of the calculations used in the blue/green question, and has many other uses.
Here are two equivalent ways to describe conditional probability. First, assume we are keeping track of a set of "equally likely outcomes," like in the original definition of probability. Then *the conditional probability of A, conditioned on B*, written $\mathsf{Pr}[A \mid B]$, is defined as

$$\mathsf{Pr}[A \mid B] = \frac{\text{the number of outcomes where } A \text{ and } B \text{ both happened}}{\text{the number of outcomes where } B \text{ happened.}}.$$

It means, *if a random experiment causes B to be true, what fraction of the time will A also be true?* For example, let X be the number on the top of a die. Then

$$\mathsf{Pr}[X \text{ is prime} \mid X \text{ is a non-square number}] = \frac{3}{4}$$

because X can be a non-square number in 4 ways (2, 3, 5, 6), and a non-square prime number in 3 ways (2, 3, 5). What is

$$\mathsf{Pr}[X \text{ is a non-square number} \mid X \text{ is prime}]?$$

An equivalent way to define this is,

$$\mathsf{Pr}[A \mid B] = \frac{\mathsf{Pr}[A \text{ and } B]}{\mathsf{Pr}[B]}.$$

Can you prove the two definitions are equivalent?

Two classical problem involving conditional probabilities

Q. You flip three coins in the air. If all three are the same (all heads or all tails) you win \$1 from me, otherwise you lose \$1 to me. I argue that this is a fair game, for the following reason: we know two of the coins are guaranteed to be identical, so there is a 50% chance that the third coin is too, thus you're just as likely to win as to lose. Am I telling the truth?

When we discuss the solution, we'll write down a way to calculate $\mathsf{Pr}[A]$ from $\mathsf{Pr}[B], \mathsf{Pr}[A \mid B]$, and $\mathsf{Pr}[A \mid \text{not } B]$. It is called *the law of total probability (LOTP)*.

Q. A cat had two kittens. If one of the kittens was female, what's the probability that both kittens were female?

Back to Probabilia

We will assume that the robber was a random person from the population, and therefore has 90% probability of being green, and 10% probability of being blue. What is the probability that the robber was green and Officer Ophelia saw them as blue? What is the probability that the robber was blue and Officer Ophelia saw them as blue? Finally, what is the probability that the robber was actually blue (like Bob), conditioned on the robber being seen as blue?

More Problems

1. A cat had a black kitten and an orange kitten. The black kitten was male. What's the probability that the orange kitten was male?

2. If we both pick random numbers from 1 to 10 (each one with probability 1/10), what is the probability yours is higher than mine? That they are equal?

3. I have three red M&Ms and two blue ones. I eat one at random and give two random ones to you. If you get at least one red M&M, what is the probability I ate a blue one? Is it higher than 2/5?

4. If you draw two cards from a standard deck of 52 cards, what's $\mathsf{Pr}[$1st card is an ace$]$? $\mathsf{Pr}[$2nd card is an ace$]$? $\mathsf{Pr}[$2nd card is an ace $\mid$ 1st card is an ace$]$? $\mathsf{Pr}[$both cards are aces$]$?

5. **Let's make a deal.** You are on a game show where there are three shiny boxes. The host, Guy Smiley, shows you that one box contains a gold coin, and the two other boxes contain turnips. (Assume you do not want a turnip, even though they are very nutritious.) The boxes are then closed and shuffled randomly. The rules say first that you pick one box, without opening it. Guy Smiley will then open a *different* box, and in particular he always opens a box containing a turnip, and eats it. That leaves two boxes: the one you picked, and another one you didn't pick. Finally, you have the choice to keep your originally chosen box, or switch to the other unopened box. If the one you end up with has the gold coin you keep it!

   Is it better to keep your original box, or switch boxes? Hint: even though you have 2 choices and you don't know which one has the turnip, it is not true that both boxes are 50% likely to contain the gold coin.

6. Assume $\mathsf{Pr}[A]$ is not 0 or 1. Show that the following three statements are either all true, or all false: $\mathsf{Pr}[A \mid B] > \mathsf{Pr}[A], \mathsf{Pr}[B \mid A] > \mathsf{Pr}[B], \mathsf{Pr}[A \mid B] > \mathsf{Pr}[A \mid \text{not } B]$. We say that events A and B are *positively correlated* when any of the above inequalities are true. A major component of statistics looks at correlations (which is different from causation).

7. It turns out YOU were arrested for being a robber. But the prince has made you an offer. He gave you two bags, 10 skull coins, and 10 happy face coins. First, you get to arrange the coins in the bags however you like; each coin must be in a bag. Second, a random bag will be selected. Third, a random coin from that bag will be selected. If it is a skull you go to jail, but if it is a happy face you go free. How can you minimize your probability of going to jail? (Hint: it is much less than 1/2.)

8. Which of these three modified dice is the best, when they are compared head-to-head and the larger number wins? Die A has values $\{2, 2, 4, 4, 9, 9\}$ printed on its sides. Die B has sides $\{1, 1, 6, 6, 8, 8\}$. Die C has sides $\{3, 3, 5, 5, 7, 7\}$.

9. **Law and Politics, Part II.** The country of Probabilia has a population with more men than women. The president has declared that a new policy will be started in order to increase the number of female births:

   Policy : When a family has a son, they are not allowed to have any more children.

   The President thinks this is a good idea because then you can have some families with several daughters, but every family will have at most one son. Will it actually help achieve the goal of having more daughters born overall than sons?