Intermediate Math Circles
March 7, 2012
Problem Set: Linear Diophantine Equations II
Solutions

1. Alyssa has a lot of mail to send. She wishes to spend exactly \$100 buying 49-cent and 53-cent stamps.
   Parts (a)-(d), will step you through determining the number of ways she can do so.
   Let's let x represent the number of 49-cent stamps she buys and y represent the number of 53-cent stamps she buys.

   a) Use the Euclidean Algorithm to calculate $\gcd(49, 53)$.

      Solution:

$$53 = 1 \cdot 49 + 4 \quad (1)$$
$$49 = 12 \cdot 4 + 1 \quad (2)$$
$$4 = 4 \cdot 1 + 0 \quad (3)$$

      So $\gcd(49,\ 53) = 1$

   b) Find a solution to $49x + 53y = 1$.

      Solution:

      Working backwards through the steps in the Euclidean Algorithm in part (a):

$$\begin{aligned} 1 &= \underline{49} - 12 \cdot \underline{4} \qquad \text{from (2)} \\ &= \underline{49} - 12 \cdot (\underline{53} - 1 \cdot \underline{49}) \qquad \text{from (1)} \\ &= 13 \cdot \underline{49} - 12 \cdot \underline{53} \end{aligned}$$

      So one solution is $x = 13$, $y = -12$.

c) Find all solutions to $49x + 53y = 10000$.

Solution:

Since $\gcd(49, 53) = 1$ and 1 divides 10000, we know that there is a solution to this linear Diophantine equation.

We already know that $$49(13) + 53(-12) = 1$$

Multiplying both sides by 10000: $$10000(49(13) + 53(-12)) = 10000(1)$$

And so $$49(10000(13)) + 53(10000(-12)) = 10000$$

Therefore, $$49(130000) + 53(-120000) = 10000$$

So one solution is $x = 130000, \; y = -120000$.

The complete solution is:
$x = 130000 + 53n, \; y = -120000 - 49n$, where n is any integer.

d) Find all solutions to $49x + 53y = 10000$, where $x \geq 0$, $y \geq 0$.

Solution:

In order to have $x \geq 0$, we would need $x = 130000 + 53n \geq 0$
And so, $53n \geq -130000$
Thus, $n \geq -2452.83...$
and since n is an integer, this implies that we must have $n \geq -2452$.

In order to have $y \geq 0$, we would need $y = -120000 - 49n \geq 0$
And so, $49n \leq -120000$
Thus, $n \leq -2448.98...$
and since n is an integer, this implies that we must have $n \leq -2449$.

So $-2452 \leq n \leq -2449$.
So the solutions are:
$n = -2452: \; x = 44, \; y = 148$
$n = -2451: \; x = 97, \; y = 99$
$n = -2450: \; x = 150, \; y = 50$
$n = -2449: \; x = 203, \; y = 1$

e) List all of the ways that Alyssa can spend exactly \$100 buying 49-cent and 53-cent stamps.

Solution:

Alyssa has 4 possibilities. She could buy:
44 49-cent stamps and 148 53-cent stamps, or
97 49-cent stamps and 99 53-cent stamps, or
150 49-cent stamps and 50 53-cent stamps, or
203 49-cent stamps and 1 53-cent stamp

2. At a museum, an adult ticket costs \$10 and a student ticket costs \$6. Suppose your school group spends exactly \$156 on tickets for a field trip to the museum. Determine all possibilities for the number of adults and students who went on the field trip.

Solution:

Let x represent the number of student tickets and y represent the number of adult tickets. We need to solve the linear Diophantine equation $6x + 10y = 156$.

From the Euclidean Algorithm, we know:

$$10 = 1 \cdot 6 + 4 \quad (1)$$
$$6 = 1 \cdot 4 + 2 \quad (2)$$
$$4 = 2 \cdot 2 + 0 \quad (3)$$

Working backwards we get:

$$\begin{aligned} 2 &= \underline{6} - 1 \cdot \underline{4} \qquad \text{from (2)} \\ &= \underline{6} - 1 \cdot (\underline{10} - 1 \cdot \underline{6}) \qquad \text{from (1)} \\ &= 2 \cdot \underline{6} - 1 \cdot \underline{10} \end{aligned}$$

So $6(2) + 10(-1) = 2$.
But we need to solve $6x + 10y = 156$.

We already know that $6(2) + 10(-1) = 2$

Multiplying both sides by 78: $78(6(2) + 10(-1)) = 78(2)$

And so $6(78(2)) + 10(78(-1)) = 156$

Therefore, $6(156) + 10(-78) = 156$

So one solution to $6x + 10y = 156$ is $x = 156, \ y = -78$.
Therefore, the complete solution is
$x = 156 + (\frac{10}{2})n = 156 + 5n$ and $y = -78 - (\frac{6}{2})n = -78 - 3n$

Since x and y represent the number of students and adults, we need $x \geq 0$ and $y \geq 0$.

In order to have $x \geq 0$, we would need $x = 156 + 5n \geq 0$
And so, $5n \geq -156$, thus $n \geq -31.2$
and since n is an integer, this implies that we must have $n \geq -31$.

In order to have $y \geq 0$, we would need $y = -78 - 3n \geq 0$
And so, $3n \leq -78$, thus $n \leq -26$

So $-31 \leq n \leq -26$ and the possibilities are:
$n = -26: \ x = 26, \ y = 0$ (so, 26 student tickets and 0 adult tickets)
$n = -27: \ x = 21, \ y = 3$ (so, 21 student tickets and 3 adult tickets)
$n = -28: \ x = 16, \ y = 6$ (so, 16 student tickets and 6 adult tickets)
$n = -29: \ x = 11, \ y = 9$ (so, 11 student tickets and 9 adult tickets)
$n = -30: \ x = 6, \ y = 12$ (so, 6 student tickets and 12 adult tickets)
$n = -31: \ x = 1, \ y = 15$ (so, 1 student ticket and 15 adult tickets)

3. Determine the number of ways you can make exactly \$200 using exactly 1000 coins if each coin is a quarter, a dime or a nickel.

Solution:

Let q represent the number of quarters, d represent the number of dimes and n represent the number of nickels.

The information given in the problem tells us that

$$q + d + n = 1000 \quad (1)$$
$$25q + 10d + 5n = 20000, \quad \text{dividing by 5}, \quad 5q + 2d + n = 4000 \quad (2)$$

Subtracting (1) from (2) we get: $4q + d = 3000$.
So we need to solve the linear Diophantine equation $4q + d = 3000$.

$\gcd(4, 1) = 1$ and $4(0) + 1 = 1$.
Multiplying both sides by 3000: $4(0) + 1(3000) = 3000$

So one solution to $4q + d = 3000$ is $q = 0, \ d = 3000$.

Therefore, the complete solution is
$q = (\frac{1}{1})k = k$ and $d = 3000 - (\frac{4}{1})k = 3000 - 4k$

Since $q + d + n = 1000$, we have $n = 1000 - q - d = 1000 - k - (3000 - 4k) = 3k - 2000$
Since q, d and n represent coins, we need $q \geq 0$, $d \geq 0$ and $n \geq 0$.

In order to have $q \geq 0$, we would need $q = k \geq 0$

In order to have $d \geq 0$, we would need $d = 3000 - 4k \geq 0$
And so, $4k \leq 3000$, thus $k \leq 750$.

In order to have $n \geq 0$, we would need $n = 3k - 2000 \geq 0$
And so, $3k \geq 2000$, thus $k \geq 666.\bar{6}$
And since k must be an integer, we need $k \geq 667$.

So $667 \leq k \leq 750$ and so there are $750 - 667 + 1 = 84$ ways to make exactly \$200 using exactly 1000 coins if each is a quarter, dime or nickel.

4. Find the smallest positive integer x so that $157x$ leaves remainder 10 when divided by 24.

Solution:

We need to solve $157x = 24y + 10$ for integers x and y.
In other words, we need to solve the linear Diophantine equation $157x - 24y = 10$.
From the Euclidean Algorithm, we know:

$$157 = 6 \cdot 24 + 13 \quad (1)$$
$$24 = 1 \cdot 13 + 11 \quad (2)$$
$$13 = 1 \cdot 11 + 2 \quad (3)$$
$$11 = 5 \cdot 2 + 1 \quad (4)$$
$$2 = 2 \cdot 1 + 0 \quad (5)$$

Working backwards we get:

$$\begin{aligned}
1 &= \underline{11} - 5 \cdot \underline{2} && \text{from (4)} \\
&= \underline{11} - 5 \cdot (\underline{13} - 1 \cdot \underline{11}) && \text{from (3)} \\
&= 6 \cdot \underline{11} - 5 \cdot \underline{13} \\
&= 6 \cdot (\underline{24} - 1 \cdot \underline{13}) - 5 \cdot \underline{13} && \text{from (2)} \\
&= 6 \cdot \underline{24} - 11 \cdot \underline{13} \\
&= 6 \cdot \underline{24} - 11 \cdot (\underline{157} - 6 \cdot \underline{24}) && \text{from (1)} \\
&= 72 \cdot \underline{24} - 11 \cdot \underline{157}
\end{aligned}$$

So $157(-11) - 24(-72) = 1$.
But we need to solve $157x - 24y = 10$.

We already know that $\quad 157(-11) - 24(-72) = 1$

Multiplying both sides by 10: $\quad 10(157(-11) - 24(-72)) = 10(1)$

And so $\quad 157(10(-11)) - 24(10(-72)) = 10$

Therefore, $\quad 157(-110) - 24(-720) = 10$

So one solution to $157x - 24y = 10$ is $x = -110,\ \ y = -720$.
Therefore, the complete solution is
$x = -110 + (\frac{-24}{1})n = -110 - 24n$ and $y = -720 - (\frac{157}{1})n = -720 - 157n$

We are asked for the smallest positive value of x. In order to have $x \geq 0$, we would need $x = -110 - 24n \geq 0$
And so, $24n \leq -110$, thus $n \leq -4.583...$
and since n is an integer, this implies that we must have $n \leq -5$.

So the smallest possible value for x is when $n = -5$, and we have $x = -110 - 24(-5) = 10$.

5. A person cashes a cheque at the bank. By mistake the teller pays the person the number of cents as dollars and the number of dollars as cents. The person spends \$3.50 before noticing the mistake, then after counting the money finds that there is exactly double the amount of the cheque. For what amount was the cheque drawn?

Solution:

Let the original cheque be for x dollars and y cents.
The information given in the question tells us that $2(100x + y) = 100y + x - 350$.
Rearranging, we get $199x - 98y = -350$.
In other words we need to solve the linear Diophantine equation $98y - 199x = 350$.
From the Euclidean Algorithm, we know:

$$199 = 2 \cdot 98 + 3 \quad (1)$$
$$98 = 32 \cdot 3 + 2 \quad (2)$$
$$3 = 1 \cdot 2 + 1 \quad (3)$$
$$2 = 2 \cdot 1 + 0 \quad (4)$$

Working backwards we get:

$$\begin{aligned} 1 &= \underline{3} - 1 \cdot \underline{2} && \text{from (3)} \\ &= \underline{3} - 1 \cdot (\underline{98} - 32 \cdot \underline{3}) && \text{from (2)} \\ &= 33 \cdot \underline{3} - 1 \cdot \underline{98} \\ &= 33 \cdot (\underline{199} - 2 \cdot \underline{98}) - 1 \cdot \underline{98} && \text{from (1)} \\ &= 33 \cdot \underline{199} - 67 \cdot \underline{98} \end{aligned}$$

So $98(-67) - 199(-33) = 1$. But we need to solve $98y - 199x = 350$.

We already know that $\quad 98(-67) - 199(-33) = 1$

Multiplying both sides by 350: $\quad 350(98(-67) - 199(-33)) = 350(1)$

And so $\quad 98(350(-67)) - 199(350(-33)) = 350$

Therefore, $\quad 98(-23450) - 199(-11550) = 350$

So one solution to $98y - 199x = 350$ is $x = -11550, \ y = -23450$.
Therefore, the complete solution is
$x = -11550 + (\frac{98}{1})n = -11550 + 98n$ and $y = -23450 - (\frac{-199}{1})n = -23450 + 199n$

Since x and y represent the number of dollars and cents, we need $x \geq 0$ and $y \geq 0$.

In order to have $x \geq 0$, we need $x = -11550 + 98n \geq 0$, and so $98n \geq 11550$, thus $n \geq 117.86....$ Since n is an integer, this implies that we must have $n \geq 118$.

In order to have $y \geq 0$, we need $y = -23450 + 199n \geq 0$, and so $199n \geq 23450$, thus $n \geq 117.84....$ Since n is an integer, this implies that we must have $n \geq 118$.

Also, since y represents the number of cents, we need $y \leq 99$.

In order to have $y \leq 99$, we need $y = -23450 + 199n \leq 99$, and so $199n \leq 23549$, thus $n \leq 118.34....$ Since n is an integer, this implies that we must have $n \leq 118$.

So $118 \leq n \leq 118$, and so the only possibility is for $n = 118$.
When $n = 118: \ x = -11550 + 98(118) = 14, \ y = -23450 + 199(118) = 32$.
So the original cheque was for \$14.32.