Intermediate Math Circles
February 8, 2012
Contest Preparation I

Answers to Problem Sets 1 and 2

Answers to Problem Set 1:

1. D	2. A	3. D	4. B
5. D	6. D	7. E	8. D
9. E	10. B	11. E	12. D

Answers to Problem Set 2:

1. B	2. E	3. B	4. E
5. D	6. E	7. D	8. E
9. D	10. D	11. A	12. C
13. C	14. E	15. A	

** **Important:** Solutions presented here are solutions to the problems whose solutions cannot be found online at the CEMC website. If the source of a problem is provided (i.e. the contest it was taken from), a complete solution can be found online in the past contest solutions found on the CEMC website.

Problem Set 1 Additional Solutions

6. In order for $7k52$ to be divisible by 12, it must be divisible by 4 and 3. Since the last two digits of $7k52$ are divisible by 4, the entire number is divisible by 4. In order for $7k52$ to be divisible by 3, the sum of the digits must be divisible by 3. The sum of the digits is $14 + k$. The values of k can be between 0 to 9.

   When $k = 1$, the sum of the digits is 15 which is divisible by 3. Then 7152 is divisible by 3.

   When $k = 4$, the sum of the digits is 18 which is divisible by 3. Then 7452 is divisible by 3.

   When $k = 7$. the sum of the digits is 21 which is divisible by 3. Then 7752 is divisible by 3.

   So there are 3 different values of k for which $7k52$ is divisible by 12 and the answer is D.

Problem Set 2 Additional Solutions

1. The next number after 2722 that has three digits with the same number is 2777. So the least number of kilometers that must be travelled to reach a number with three digits of the same number is $2777 - 2722 = 55$. Since 55 is between 50 and 100, the answer is B.

3. To find the value of $0 * 0$, substitute 0 into the definition of $a * b$ for a and b.

$$\begin{aligned} a * b &= (a+1)(b-1) \\ 0 * 0 &= (0+1)(0-1) \\ &= (1) \times (-1) \\ &= -1 \end{aligned}$$

   Therefore the answer is B.

5. We see that 7 is a divisor of 777. Every time we subtract 7, the resulting number will also be divisible by 7. (If $a \div b = q$, then $(a-b) \div b = (a \div b) - (b \div b) = q - 1$. So $a - b$ is also divisible by b.) From the numbers in the list of possible answers, only 42 is divisible by 7. Therefore the answer is D.

6. The total cost of the cost is $\$8.43 + \$13.37 + \$2.46 = \$24.26$.

   If Pat and Chris split the cost of the dinner equally, the cost of one share of the dinner is $\$24.26 \div 2 = \$12.13$. But since Pat has already paid \$2.46, he owes Chris
   $\$12.13 - \$2.46 = \$9.67$. Thus, the answer is E.

7. **Solution 1:** Let x represent the value of the smallest integer. Since the integers are consecutive, we can represent the other four integers by $(x+1), (x+2), (x+3)$, and $(x+4)$. Then the sum of the integers is:

$$\begin{aligned} x + (x+1) + (x+2) + (x+3) + (x+4) &= 75 \\ 5x + 10 &= 75 \\ 5x &= 65 \\ x &= 13 \end{aligned}$$

   Therefore, the value of the smallest integer is 13 and the value of the largest integer is $13 + 4 = 17$. It follows that the sum of the smallest and largest integers is $13 + 17 = 30$ and the answer is D.

7. **Solution 2:** Let x represent the value of the middle integer. Since the integers are consecutive, we can represent the other four integers by $(x-1), (x-2), (x+1)$, and $(x+2)$. Then the sum of the integers is:

$$\begin{aligned} (x-2)+(x-1)+x+(x+1)+(x+2) &= 75 \\ 5x &= 75 \\ x &= 15 \end{aligned}$$

The sum of the smallest and largest integers is $(x-2)+(x+2)=2x=30$ and the answer is D.

8. Let $x_1,\ x_2,\ x_3,\ x_4$ represent the number of points Megan scored in the first, second, third and fourth games, respectively.

After the first three games Megan scored an average of 18 points per game so $\frac{x_1+x_2+x_3}{3}=18$ and $x_1+x_2+x_3=54$ follows.

After the fourth game her points per game average drops to 17. So, $\frac{x_1+x_2+x_3+x_4}{4}=17$ and $x_1+x_2+x_3+x_4=68$ follows. (1)

But $x_1+x_2+x_3=54$. We can substitute this into equation (1) obtaining $54+x_4=68$ and $x_4=14$ follows.

Therefore, Megan scored 14 points in the fourth game and the answer is E.

10. The smallest perfect square over 10 is $4^2=16$ and the largest perfect square under 200 is $14^2=196$. For any odd number between 4 and 14, its square is also odd. Then its square increased by 1 is even and therefore not prime. This leaves the even numbers from 4 to 14, namely $4, 6, 8, 10, 12,$ and 14, to consider. Checking gives us the following

$$\begin{array}{rcll} 4^2=16 & \text{and} & 4^2+1=17 & \text{which is prime} \\ 6^2=36 & \text{and} & 6^2+1=37 & \text{which is prime} \\ 8^2=64 & \text{and} & 8^2+1=65 & \text{which is } \textbf{not} \text{ prime} \\ 10^2=100 & \text{and} & 10^2+1=101 & \text{which is prime} \\ 12^2=144 & \text{and} & 12^2+1=145 & \text{which is } \textbf{not} \text{ prime} \\ 14^2=196 & \text{and} & 14^2+1=197 & \text{which is prime} \end{array}$$

These are the only four possible values for n and it follows that the answer is D.

In general, it is a very difficult problem to determine whether a very large number is prime or not. But for small numbers, we can determine if a number is prime by dividing by **every prime** less than its **square root**. If none of these primes divides the number, then the number is prime.