University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

# Grade 6 Math Circles
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# *Probability of Games*

"Gambling is the wagering of money or something of material value on an event with an uncertain outcome with the primary intent of winning additional money and/or material goods. " (Wikipedia definition)

In other words, gambling is when you put something at stake, not sure if you'll win more back or lose what you put in.

Some of these children's gambling/ arcade places include: the arcade section in movie theatres, the Great Canadian Midway at Clifton Hill, Niagara Falls, Playdium Mississauga, and many more.

Gambling isn't so bad if you have discretion and know when to stop. In other words, you are just doing it for some fun.

Gambling is BAD if it becomes your life and you start throwing money with the mindset that "I'm gonna get it back for sure this time!"

In fact, for most people, they lose a lot more than what the put in when they gamble.

Hence, think twice before you wager that precious allowance of yours.

In this lesson, I will show you WHY most people CANNOT make a living out of wagering money in games.

## Definitions:

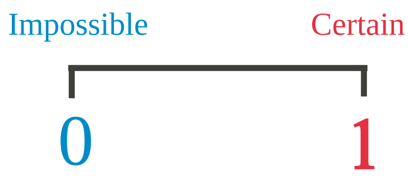
**Probability** is the study of "how likely will my desired outcome occur in an experiment?"

An **experiment** can refer to any well-defined process in which observations are made, for example, observing the face that turns up on a die toss is an experiment.

**Expected gain** in our case refers to "taking into consideration the probability of every possible outcome, how much will I gain from my wager?"

Before we start, we must understand that:

**If an outcome definitely will not occur, then it has probability 0.**
**If an outcome definitely will occur, then it has probability 1.**
$P$**(outcome A occurs)**$+P$**(outcome A doesn't occur)**$=$ **1.**




## Probability Equation:

Probability can be very easy:

### Example:

1. In a bingo machine with these numbers: $1, 21, 5, 3, 9, 16$ the probability I get a 15 is? 0
2. In my drawer of red socks, the probability that I pick out a red sock is? 1
3. In a game of coin toss, the probability that I guessed correctly on a toss is? $\frac{1}{2}$
4. The probability that I throw a fair die and it doesn't land on 2 is? $\frac{5}{6}$
5. This seems like a pretty fair way to earn some extra allowance for the week: A machine have 3 cards, red, blue and black in my pouch. If I draw the red card out, I get \$2 back. It costs me \$1 to play each time. Do I end up getting more money than I put in in the long run? No, you lose money in the long run, approximately you lose \$0.33 each game.

The probability equation is calculated as follows:

$$P(\textbf{my desired outcome occurs}) = \frac{\textbf{\# of desired outcomes}}{\textbf{Total \# of outcomes}}$$

This method of determining a probability is based on the assumption that all outcomes are equally likely.

## Expected Gain Equation:

### Example:

1. A random number generator generates either $\{1, 2, 3\}$, with you earning the same amount of dollars as the number generator generates. It costs \$1 per play. What is your expected gain per play? Expected gain: $\frac{1}{3} \times 1 + \frac{1}{3} \times 2 + \frac{1}{3} \times 3 - 1 = \$1.00$. You win about \$1.00 each game.

2. Same random generator as above. Except now that it costs \$2.50 per play, what is your expected gain per play? You lose about \$0.50 each game.

The expected gain when an experiment is performed is calculated as follows:

Let $x$ be the value gained from a certain outcome; $P(x)$ be the probability that this outcome occurs.

$$\textbf{Expected gain}=(\sum xP(x)) - \textbf{cost of experiment}$$

In another word, the expected gain is when you add up the value gained from all possible outcomes, taking into consideration the probability of each outcome's occurrence, then subtract how much you payed to play the game.

## Idea of Dependent Events and Conditional Probability Equation:

When trying to determine the probability of an outcome given the occurrence of some other outcome, we call this conditional probability:

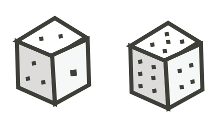
**Example:**

1. Suppose there are 4 holes on an arcade machine, and they have values {1, 10, 50, 100}. You launch 2 balls one by one, the first hole that your first ball lands in closes when you launch your 2nd ball. Supposed that your first ball landed in 100, what is the probability that your second ball will land in the 50 hole? $\frac{1}{3}$

If this helps you at all, conditional probability is calculated s follows:

$$\textbf{P(Given outcome A occurred, outcome B will occur)}=\frac{\textbf{P(outcome A and outcome B occurs)}}{\textbf{P(outcome A occurs)}}$$

Good! Let's move on to some more interesting stuff. After all, this lesson is about teaching you how the gambling game makers always make money off you.



## The Die has many faces:

1. A simple game often played on a betting table is "Under and Over 7". You take 2 dice, and you throw them, then you guess whether your roll is {under 7, over 7, or exactly 7}. If you get it right for {under 7 or over 7}, you get back twice your wager. If you get it right for {exactly 7}, you get back 5 times your wager. You want to make some big bucks, so you guess that it's exactly 7 each time. You play for 20 games, wager \$2 each time, what is your expected gain?

There are 36 possible combinations with the 2 dices. 6 of them gives you the sum of 7. Namely: {1,6},{6,1},{2,5},{5,2},{3,4},{4,3}. Expected gain: $\frac{6}{36} \times 10 - 2 = \frac{60}{36} - \frac{72}{36} = -\frac{12}{36} = -\frac{1}{3} \approx -\$0.33$. You lose on average \$0.33 each time.

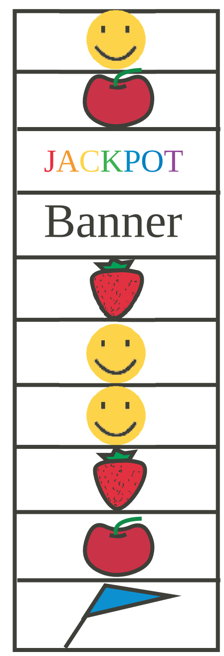
2. Same game as above, this time you've learned your lesson and decides to shoot for the smaller win. So you guess either under 7 or over 7 each time and never exactly 7. You again play 20 games, wager \$2 each time, what is your expected gain?

   The probability of having over 7 or under 7 is $\frac{15}{36}$ (try to figure this out). Expected gain: $\frac{15}{36} \times 4 - 2 = \frac{60}{36} - 2 \approx -\$0.33$. Again, I lose \$0.33 on average.

3. In a dice betting game, you will throw a die twice. You will only win if each time you toss the die, it lands on 4. What is the probability that you win?
   The probability that I win is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

## The Token swallowing machines:

1. Slot machines! The most popular token eater in casinos and midways. The are 3 rolling panels, you throw in your coin to get the panels rolling. You press the big shiny red button once, and the 3 panels stop one by one, can't get any simpler. A slot machine has a panel that looks like this. It costs me 1 token to spin the machine each time.




   (a) What is the probability that I land the jack pot? $\frac{1}{9} \times \frac{1}{9} \times \frac{1}{9} = \frac{1}{729}$

   (b) What is the probability that I get 3 happy faces? $\frac{3}{9} \times \frac{3}{9} \times \frac{3}{9} = \frac{1}{27}$

   (c) What is the probability that I get 3 banners? $\frac{1}{9} \times \frac{1}{9} \times \frac{1}{9} = \frac{1}{729}$

   (d) What is the probability that I get 2 of jack pot?

   There are 3 ways of getting 2 jackpots. Namely, the first two, the first and the third and the second and the third. Each of these combinations have probability $3 \times (\frac{1}{9} \times \frac{1}{9}) = \frac{3}{81}$.

   (e) What is the probability that I win anything at all? (3 of anything or 2 of jackpot is considered a win).

   On the panel, there are strawberries, apples and flag that we haven't calculated. There are 2 strawberries and 2 apples and 1 flag. I have the same probability of getting 3 flags as the jackpot or the banner. For strawberries and apples, I have a probability of $\frac{2}{9} \times \frac{2}{9} \times \frac{2}{9} = \frac{8}{729}$. Therefore, I have a probability of total: $3 \times \frac{1}{729} + 2 \times \frac{8}{729} + \frac{27}{729} + \frac{3}{81} = \frac{45}{729} \approx 6.2\%$ chance of winning anything.

   (f) For the jackpot, I get 50 tickets back. For 3 banners, I get 20 tickets back. For 3 of anything else I get 10 tickets back. For 2 of jackpot, I get 10 tickets back. Otherwise I get 1 ticket back. How much ticket do I expect returned from each round of playing slot machine? How many tickets do I expect to have after I spent 100 tokens in it? If a token is worth 5 tickets, is this game worth spending token in?

   Each round of playing, I expect this much ticket: $\frac{1}{729} \times 50 + \frac{1}{729} \times 20 + 2 \times \frac{8}{729} \times 10 + \frac{27}{729} \times 10 + \frac{3}{81} \times 10 = \frac{770}{729} \approx 1.05$ tickets.

   After I spend 100 tokens in it, I expect to get back about $100 \times 1.05 = 105$ tickets back.

   100 tokens is equivalent to 500 tickets. No this game is not worth spending tokens in.

## Others:

1. Tim's RRRRRRoll Up The Rim!!! Are we addicted to this stuff yet?
   Every year around spring time, Tim Hortons' produces a fascinating little Rrrrrroll-up-the-rim contest where after you buy a hot beverage and drank it, you roll up the rim and check if you've won any prizes. (Apparently this jacks up Tim Hortons' sale by quite a bit during the months of play)

   As of year 2012, the information they give you on the cup is that there is a 1 in 6 chance of winning some prize.
   The prizes for 2012 are:

   - 47,000,000 food prizes
   - 25,000 $100 Tim's card
   - 5,000 Panasonic Digital Camera
   - 1,000 Camping Packages
   - 100 Panasonic 3D TV Packages, and last but not least...
   - 40 TOYOTA CAMRY

   (a) How many cups am I expected to buy to win some kind of prize? 6

   (b) How many Rrrroll Up the Rim cups are distributed approximately? $(47000000+25000+5000+1000+100+40) \times 6 = 282186840$ cups

   (c) I really like that camera. What is the probability that I get a camera? Approximately 1 in 57000.

   (d) How many cups am I expected to buy to win that camera? 57000.

   (e) What is the probability that I get a car? Approximately 1 in 7000000.

   (f) How many cups am I expected to buy to win a car? Approximately 7000000.

   (g) If each cup costs around $1.50 and I buy until I win a car, how much would I have spent on Timmie's hot beverages? The car's market value is approximately $35,000. Is this worth it? Approximately 10 million and more. No clearly this is not really worth it. However, you do get the value of the coffee, so it might be worth it if you drink coffee on regular basis and do not just get coffee for roll up the rim purpose only.

   (h) If each cup of hot beverage (coffee, tea, french vanilla, hot chocolate, etc.) costs you about $1.50 on average, and you buy about 2 cup per day hoping for good luck. Suppose the contest run until end of April for approximately 60 days:

      i. How many cup of Tim's hot beverages would you've bought over 60 days? 120

      ii. How much would you spend on Timmie's over 60 days of the contest? $180

      iii. How many food prizes are you expected to win approximately? $120 \times \frac{1}{6} = 20$

2. McDonald's Happy Meal Toys!!! We all love these! You receive a random toy each time you buy a happy meal. This new set that just came out has 6 toys.

   (a) What is the probability that you bought 6 happy meals and you manage to collect the entire set?

   $\frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} \times \frac{1}{6} = \frac{120}{6^6} \approx 0.26\%$

(b) *How many happy meals are you expected to buy before you can collect all 6?

$\frac{6}{6}+\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+\frac{6}{1}\approx 14.7\approx 15$. That is, you have to buy approximately 15 happy meals to collect all 6 toys.

(c) Let's look at the following:

i. A pack of about 40 chicken nuggets costs around \$10

ii. A package of about 10 apple (the kind they give you in happy meals) costs about \$4

iii. A bottle of sweet and sour sauce (200ml) costs about \$3

iv. A carton of milk, 2 Litres, costs about \$5.50

v. Highlight! The toys are "Made in China" and costs, believe it or not, about \$0.10! (10 cents each!)

Now, in a Happy Meal, you get 4 pieces of Chicken McNuggets, $\frac{1}{2}$ apple, 10ml Sweet and Sour Sauce, and approximately 200ml of milk. Oh! Let's not forget about the 10 cents toy! If Mcdonald's sells the Chicken McNugget Happy Meal for \$6, how much money do they make off you over the course that you collect all 6 toys?

The actual material cost for each happy meal at McDonald's is $1+0.2+0.15+0.55+0.1=2$. Total cost: over 15 such meals is $\$2\times 15=\$30$. Therefore, McDonald's make approximately $\$4\times 15=\$60$ off you over the course of you trying to collect all 6 toys.

## The expected gain experiment on coin toss:

With a partner, grab a dime and 10 "tokens" each. One of you is "head", the other is "tail". Each round, both of you put 2 tokens into the wager pile and if it lands on head, the "head" gets pile to him/her, vice versa for tail. Whoever loses all their tokens first loses. Write down an equation for each person's expected gain per round and verify that it is 0 tokens per round.

## Exercises:


### Spinning wheels

1. They say you cut a pie into 6 slices and share it with 6 people, then everyone gets $\frac{1}{6}$ of a slice. If you have a spinning wheel divided into 6 equal pieces, and you colour one of them red, the rest green...

   (a) What is the probability that a spin lands on green?

   (b) Same spinning wheel as described above, 1 piece out of 6 is red, the other 5 pieces green. You spin the wheel 6 times, recording the colour you land on each time. What is the probability that you get exactly this pattern: {green, green, red, green, red, green} ?

   (c) Same spinning wheel as described above. You spin the wheel, if you land on green, then you keep spinning until you land on red. What is the probability that you'll spin exactly 3 times?

   (d) Same spinning wheel as above. You throw in the machine \$2 to spin it each time. If it lands on red, the machine returns out \$5, if it lands on green, the machine returns out \$1. Is this game worth playing? (Generally, a game is worth playing if you can make money from it on average and not worth playing otherwise)

2. You now have 2 spinning wheels available to you. Each of them divided into 5 equal pieces. The pieces are labelled from 1 to 5. I can guess whether the sum of the spinners will be greater than 5, exactly 5 or less than 5.

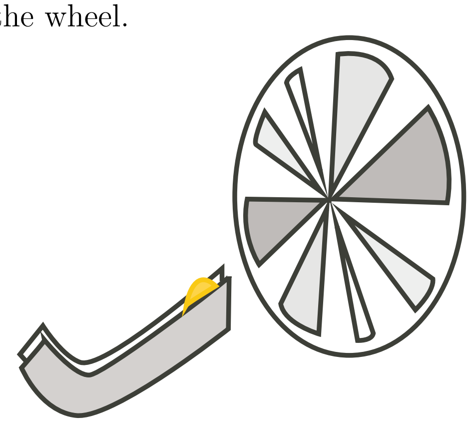
   (a) What is the probability that the sum of the spinners is exactly 5?

   (b) If you throw in \$2 to spin, and the machine will return out \$1 for a sum of either greater than or less than 5, and return out \$5 for a sum of exactly 5. Would you play this game?

3. A type of popular spinning wheel (roll-the-token-see-where-it-drops) in midways looks like this. (Insert picture here). Assuming that the white space makes up $\frac{1}{2}$ of the space on the wheel, and that if your token were to roll and hit the white space, it'll drop down to the sink and earning you nothing back. This is the architecture of the wheel.

   With respect to the ENTIRE wheel:

   - There are 2 \$1 slots, each having inside angle 50°
   - There are 2 \$2 slots, each having inside angle 20°
   - There are 2 \$5 slots, each having inside angle 15°
   - There are 2 \$20 slots, each having inside angle 5°



(a) Suppose you use the Toonie as your token. What is your expected gain on a roll? Is it worth playing?

(b) Now the organizers decide to be nice. If your first coin hits the white space, you'll receive a second chance. (However if your first coin does drop into one of the reward slots then you don't get a second go). However, on the second go, the \$1 and \$2 slots will become white spaces, but the \$5 slots will become \$20 slots and \$20 slots will become \$100 slots.

   i. Given that your first roll hit the white space and now you are on your second chance, what is the probability that your second toss will hit the white space?

   ii. You are starting a new roll, what is the probability that you'll earn the \$100?

   iii. What is your expected gain on a roll in this new version of the game? Is it worth playing now?

**<u>Cards</u>**

4. You have standard deck of 52 cards (a standard deck without the jokers). Evaluate the following probability:

   (a) You draw a card and it's a Jack of Spades?

   (b) You draw a card and it's any Diamond card?

   (c) You currently have 3 Diamonds and 1 Hearts in your hands, these were previously drawn from the deck without replacement. What is the probability you draw one more and it's a Diamond?

   (d) We'll calculate the value of a card by the following:

   - Each numbered card has the value of its number.
   - Each of the faced cards has value 1.

   i. You draw two cards without replacement, what is the probability that their sum is 18?

   ii. You currently have 3 cards in your hands, namely King of Hearts, 7 of Spades and 10 of Clubs, what is the probability that you draw one more card and the sum of the cards in your hands is less than or equal to 21?

   (e) The Queen of Spades is considered bad in some card game. What is the probability that you draw 13 cards (a hand) from the deck without replacement and you DON'T draw the Queen of Spades?

   (f) What is the probability that you draw 13 cards and you do get a Queen of Spades?

**<u>Bingo</u>**

5. Choosing your bingo card: There are numbers 1 to 50 to choose from to put onto the 25 grids of the bingo grid.

   (a) If you can repeat a number, how many bingo cards are possible (two cards that are rotations of each other are considered different)? (write down an expression, but do not evaluate)

   (b) If you cannot repeat a number, how many bingo cards are possible? (write down an expression but do not evaluate)

**Stuffed Animal Claw**

6. These things are everywhere! In the mall, a plaza, any arcade, even parks! In them are cute little stuffed animals and all you have to do is throw in a **Toonie**, aim the claw, drop and grab, hoping to get a toy that you want. Now here's the catch, most people have the tendency to go for the one they WANT as opposed to the EASIEST one to get, thus often leading to the tragic empty claw. In general, a person is likely to grab nothing $\frac{3}{4}$ times. Additionally, even when the claw does grab something, there is a $\frac{1}{2}$ probability that the toy doesn't land where you want it to. (Aw bummer!)

   (a) What is the probability that one does actually get a stuffed animal?

   (b) If a single stuffed animal is worth \$5, what is the person's expected gain on a single game?

   (c) These vendors have decided to be nice. So they've introduced these newer ones where the animals are tiny but you may play until you grab one (or more, in a single grab). These tiny little stuffed animals are, let's face it, \$1 from Dollarama. A person has $\frac{1}{10}$ chance that they'll grab 2 in a single grab. What is the person's expected gain on a single game?

   (d) Soon these vendors discovered that nobody really wants to pay \$2 for a tiny stuffed animal and that sales are dropping. So they've made a newer design. That is, your have 1 try at getting the bigger ones. Then if you don't get anything, your claw moves to the other pit where all stuffed animals are the tiny ones, and you can play until you get one (or more). With the same probability for getting big and small as described in the above parts. What is the person's expected gain on a single game?

   (e) With the same machine as described in part d). How often are you expected to get a big one? (1 in how many games?)

   (f) You played \$30 worth of game (i.e. 15 games) on the same machine described in part d) and ended up happily with a basket full of cute stuffed animals. (Quantity often makes people happy and neglect the cost). With the animals priced as described above, and can be purchased for the same price in stores, how many big ones do you expect to have in your basket, how many small ones? Is it really worth it?