UNIVERSITY OF WATERLOO
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
April 4, 2012
Math Olympics Solutions

Speedy Arithmetic:

100 points: $1469273 \times 11 =?$
Answer: 16,162,003
(It is suggested that you use the multiplication trick for 11)

200 points: How many values are there between 928 and 1024 that are divisible by 9?
Answer: 10
Here's why: The smallest number greater than 928 that is divisible by 9 is 936 (whose sum of digits is 18). Similarly, the largest number smaller than 1024 that is divisible by 9 is 1017 (whose sum of digits is 9). $1017 - 936 = 81$, $81 \div 9 = 9$. We've that 936 and 1017 are both inclusive, hence we've $9 + 1 = 10$ numbers here that are divisible by 9.

300 points: $\frac{1}{2} \times \frac{3}{2} \times \frac{12}{6} \times \frac{2}{4} \times \frac{80}{100} \times \frac{10}{4} \times \frac{1}{3} \times =?$
Answer: $\frac{1}{2}$
Here's why: Let's first reduce all the fractions to lowest terms to get our new equation: $\frac{1}{2} \times \frac{3}{2} \times \frac{2}{1} \times \frac{1}{2} \times \frac{4}{5} \times \frac{5}{2} \times \frac{1}{3}$ Then we reduce the numerator and the denominators before we multiply. We note that after crossing out all the numerators and denominators that are equal, the only fraction left is $\frac{1}{2}$. Hence that's the answer.

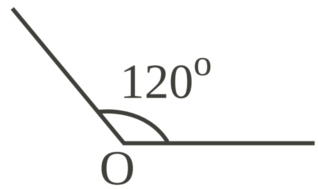
400 points: For what value(s) of $p and q$ is $2p346q$ divisible by 6?
Answer: 17
Here's why: In order for a number to divide 6, it must be divisible by both 2 and 3 (namely, the prime factors of 6). A number is divisible by 2 if and only if it is an even number. Additionally, a number is divisible by 3 if and only if the sum of its digits is a multiple of 3. So, q must be 0,2,4,6, or 8. If q is 0 or 6, then p can be 3, 6, or 9. If q is 2 or 8, then p can be 1, 4, or 7. Lastly, if q is 4, then p can be 2, 5, or 8.

500 points: Prove the divisibility rule for 3 for a 4-digit number. (i.e. A (4-digit) number is divisible by 3 if and only if the sum of its digits is divisible by 3).

<u>**Ancient Geometric Constructions:**</u>

100 points: Who is the ancient Greek mathematician is widely known as the father of geometry?
Answer: Euclid

200 points: Make a 60°angle using the diagram below:



Answer: Bisect the angle.
Here's how:

1. Center your compass at O.
2. Position the other end of your compass any where on either one of the lines, draw an arc crossing over the middle of the angle.
3. Keeping the compass width the same, still centered at O, now anchor the compass on the other line, draw another arc crossing over the middle of the angle.
4. Connect the intersection point of these two arcs with O to get your 60°angle.

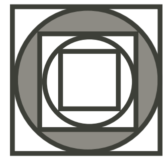
300 points: Copy this line: (the resulting copy's orientation doesn't matter)

Here's how:

1. Center your compass at O.
2. Position the other end of your compass at P.
3. Keeping the width of the compass, move the compass somewhere else on to the paper. Find a new center and draw a circle with this compass width.
4. Pick any point on the circle and connect it with your new center, this is a copy of the line OP.

400 points: With its sides being whole numbers, what is the smallest possible perimeter of a non-equilateral triangle?
Answer: 5
Here's why: Many of you I'm guess would've guessed 4 right off the bat. Side length={1,1,2}. However, this is not a triangle. The sum of the lengths of any two sides of a triangle must be longer than its 3rd side, hence the above side length does not form a triangle. Therefore, the smallest such triangle possible is one with sides {1,2,2}, and thus has perimeter 5.

500 points: The smallest square has area 32. What is the area of the shaded region in the diagram below?



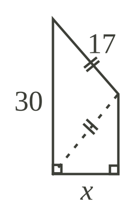
Answer: $32\pi - 64$
Here's why: The 2nd largest square has twice the area as the smallest square, thus it has area 64. (For proof of this look up the solution to the exercise on the Ancient Geometric Construction week) Then the 2nd largest square must have side length 8, thus half of its side length is 4. Then connecting from the center of these shapes to the corner of intersection of the 2nd largest square and the larger circle is the radius of the larger circle. This has length, by the *Pythagorean Theorem*, $\sqrt{4^2 + 4^2} = 4\sqrt{2}$. Then the larger circle has area $(4\sqrt{2})^2\pi = 32\pi$. The area of the shaded region= Area of larger circle− Area of 2nd largest square=$32\pi - 64$.

Algebraic Approaches:

100 points: The average of 4 numbers is 20. One of the numbers is 18. What is the sum of the other 3 numbers?
Answer: 62
Here's why: The sum of the 4 number is $20 \times 4 = 80$. One of the numbers is 18, then the sum of the other 3 numbers is $80 - 18 = 62$.

200 points: In a science experiment that involves looking at the growth of plants, we are told that plants require liquid, soil and light to grow. If you are given 5 kinds of liquid, 3 kinds of soil and 2 kinds of light, How many plants do you need to plant at minimum to guarantee that at least 2 out of all the plants grow under exactly the same conditions?
Answer: 31
Here's why: There are $5 \times 3 \times 2 = 30$ different plant conditions that my plants can grow under. By the *Pigeonhole Principle* we need at least 31 plants to ensure that 2 of them grow under the same condition.

300 points: Find the value of x.



Answer: $x = 15$
Here's why: This is a direct application of the *Pythagorean Theorem.* Half of the longer side of the trapezoid is 15. Then I have a right angled triangle with hypotenuse=17, one side=15, then the other side, by the *Pythagorean Theorem* is $\sqrt{17^2 - 15^2} = 8$.

400 points: It's spring time, and that means... STRAWBERRY PICKING! Laura can fill a basket with strawberries in 3 hours; Grace on the other hand goofs around when she picks strawberries and can fill the same kind of basket in 5 hours. How long would it take them together to fill the basket?

Answer: $\frac{15}{8}$ hours $\approx$ 1 hour 53 minutes

Here's why: Let the total amount of strawberry the basket holds be S. Then Laura picks $\frac{S}{3}$ units of strawberry each hour and Grace picks $\frac{S}{5}$ units of strawberry each hour. Together, they pick $\frac{S}{3} + \frac{S}{5} = \frac{8S}{15}$ units each hour. The total amount is S. So together, it would take them $S \div \frac{8S}{15} = \frac{15}{8}$ hours.

500 points: How many positive integers n, with $n \leq 100$, are there such that $n^3 + 7n^2$ is a *perfect square*? (A *perfect square* is a number whose square root is an integer, for example, 1, 4, 9, etc...)

Answer: 8

Here's why: We factor out the n^2 term to get the expression into $n^2(n+7)$. Then for any n, n^2 is a perfect square. Now we need to make sure $(n+7)$ is a perfect square. Since n has to be a positive integer, n must be at least 2 ($2 + 7 = 9$ is a perfect square). Also, for $n = 100$, $n + 7 = 107$ is not a perfect square. The largest perfect square that is less than 107 is $100 = 10^2$. Therefore, the largest n can be is 93. Since $9 = 3^2$ and $100 = 10^2$, then from 3 to 10 there are 8 numbers in between whose squares $n + 7$ can take on. Hence there are 8 such n.

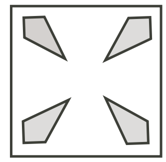
Spatial Visualization and Origami:

100 points: A fold that could be pressed flat on a surface (i.e. 2D in a strict sense) is called flat-fold. It is 2-colourable.

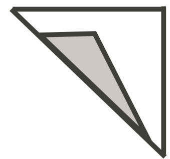
200 points: The picture below shows Sally with a very festive hair-do and make up as seen in the mirror. When Sally is facing you, what does she look like?

Answer: This is what Sally actually looks like:

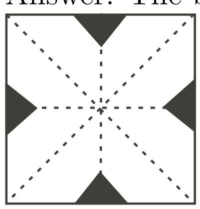
300 points: How many times does the paper need to be folded to get the pattern in the diagram below down to its *smallest component*? What does its smallest component look like? (recall that a *smallest component* is the portion of a whole pattern such that it is no longer symmetrical in any way).



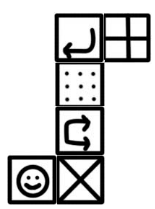
Answer: Fold 3 times. Its smallest component looks like this:

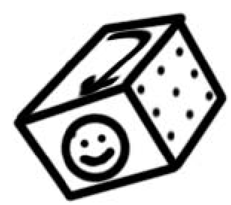


400 points: A square sheet of breakfast crepes is folded 3 times, creating an isosceles triangle each time. If a bite is taken out of the 90°angle such that the bite mark parallels the hypotenuse. What does my unfolded crepe look like?
Answer: The black portion in the diagram below is the portion that was bitten off



500 points: The diagram below is a construction diagram of a cube. After constructing, is this configuration possible?



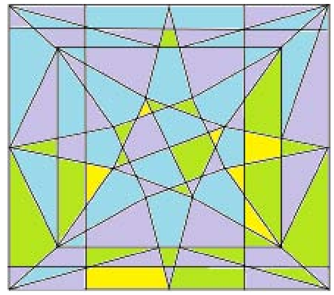


Answer: Not possible! (The happy face is upside-down!) (Try to visualize these problems without having to make the actual cube, if you feel that your brain is totally meshed up in the end, then cut out the sheet and try it out!)

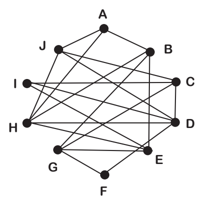
<u>Introduction to Graph Theory:</u>

100 points: Define a path and a cycle.
A **path** starts at a vertex, then travels along an edge to another vertex, and continues this pattern a finite number of times until it reaches a designated vertex, never visiting the same vertex twice. We represent the path by listing the sequence of vertices. A **cycle** is similar to a path but it starts and ends at the same vertex.

200 points: Properly colour the graph below using at most 4 colours.

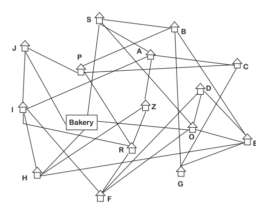


300 points: Determine a path from vertex A to F using the graph below that includes every vertex.



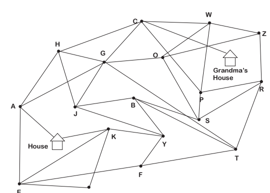
One path: A,H,B,G,E,I,C,J,D,F

400 points: Bessy the baker needs to deliver 15 cakes to 15 different houses. Using the graph below, determine a route that Bessy should take from her bakery, to each house, and return to the bakery without taking the same road or visiting the same house twice.



One cycle: Bakery, S,B,G,C,A,I,F,O,D,E,H,Z,R,P,J, Bakery

500 points: Little Red Riding Hood needs to walk through the forest to deliver a basket full of food to her Grandma's house. She has discovered that if she passes through every checkpoint (vertex) on the graph below, she is able to avoid the Big Bad Wolf. Determine a path that Red Riding Hood should take to Grandma's House without visiting the same checkpoint or walking along the same path twice.



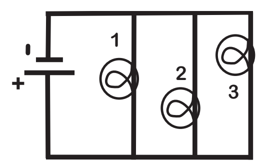
One path: House, K,D,E,F,Y,J,H,A,G,T,B,S,O,W,Z,R,P,C,Grandma's House.

Circuits:

100 points: State Ohm's law and the properties for current, voltage and resistance for series and parallel circuits.

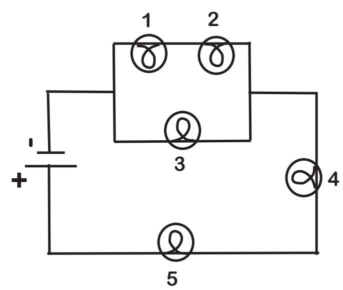
200 points: A 6V battery is connected to a single light bulb in a circuit. If the resistance at the light bulb is measured to be 2 Ω, what is the current passing through the light bulb?

300 points: For the diagram and corresponding table below, fill in the missing values in the table using Ohm's Law and circuit properties.



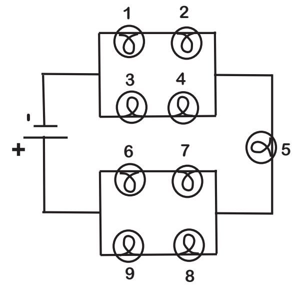
$V_{total} = 16$ V	$V_1 = 16$ V	$V_2 = 16$ V	$V_3 = 16$ V
$I_{total} = 8$ A	$I_1 = 4$ A	$I_2 = 2$A	$I_3 = 2$ A
$R_{total} = 2\Omega$	$R_1 = 4\Omega$	$R_2 = 8\Omega$	$R_3 = 8\Omega$

400 points: For the diagram and corresponding table below, fill in the missing values in the table using Ohm's Law and circuit properties.



$V_{total} = 10$ V	$V_1 = \frac{2}{3}$ V	$V_2 = \frac{4}{3}$ V	$V_3 = 2$ V	$V_4 = 3$ V	$V_5 = 5$ V
$I_{total} = 1$A	$I_1 = \frac{2}{3}$ A	$I_2 = \frac{2}{3}$A	$I_3 = \frac{1}{3}$ A	$I_4 = 1$ A	$I_5 = 1$ A
$R_{total} = 10\Omega$	$R_1 = 1\Omega$	$R_2 = 2\Omega$	$R_3 = 6\Omega$	$R_4 = 3\Omega$	$R_5 = 5\Omega$

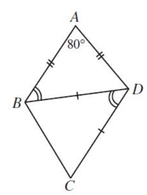
500 points: For the diagram and corresponding table below, fill in the missing values in the table using Ohm's Law and circuit properties.



$V_{total} = 36$ V	$V_1 = 8$ V	$V_2 = 4$ V	$V_3 = 8$ V	$V_4 = 4$ V	$V_5 = 6$ V
$I_{total} = 3$ A	$I_1 = 2$ A	$I_2 = 2$ A	$I_3 = 1$ A	$I_4 = 1$ A	$I_5 = 3$ A
$R_{total} = 12\Omega$	$R_1 = 4\Omega$	$R_2 = 2\Omega$	$R_3 = 8\Omega$	$R_4 = 4\Omega$	$R_5 = 2\Omega$
	$V_6 = 12$ V	$V_7 = 6$ V	$V_8 = \frac{27}{2}$ V	$V_9 = \frac{9}{2}$ V	
	$I_6 = \frac{3}{2}$ A	$I_7 = \frac{3}{2}$ A	$I_8 = \frac{3}{2}$ A	$I_9 = \frac{3}{2}$ A	
	$R_6 = 8\Omega$	$R_7 = 4\Omega$	$R_8 = 9\Omega$	$R_9 = 3\Omega$	

Gauss Contest:

100 points: In the diagram below, ∠ABD = ∠BDC and ∠DAB = 80. Also, AB = AD and DB = DC. What is the measure of ∠BCD?



Since $\triangle ABD$ is isosceles, $\angle ABD = \angle ADB = 50$ °. Then since $\angle ABD = \angle BDC$, $\angle BDC = 50°$. $triangleBDC$ is also an isosceles triangle, so $\angle DBC = \angle BCD = 65$ °

200 points: Chloe has made a code out of the alphabet by assigning a numerical value to each letter. She then assigns a numerical value to a word by adding up the numerical values of the letters in the word. Using her code, the numerical value of BAT is 6. Also, her code gives numerical values of 8 to CAT and 12 to CAR. Using her code, what is the numerical value of BAR?

$BAT = B + A + T = 6$

$B + A = 6 - T$

$CAR = C + A + R = 12$

$R = 12 - C - A$

$BAR = B + A + R = 6 - T + 12 - C - A = 18 - (C + A + T) = 18 - CAT = 18 - 8 = 10$

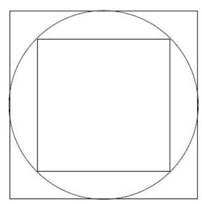
300 points: In Math Idol, there was a total of 5 219 000 votes cast for four potential Idols. The winner received 22 000 more votes than the 2nd place contestant, 30 000 more than the 3rd place contestant, and 73 000 more than the 4th place contestant. How many votes did the winner receive?

Let the total number of votes that the person in first place received be represented by x. Then the second place contestant received $x - 22000$, the third place contestant received $x - 30000$, and the fourth place contestant received $x - 73000$ votes. The total number of votes is:

$$\begin{aligned} x + x - 22000 + x - 30000 + x - 73000 &= 5219000 \\ 4x - 125000 &= 5219000 \\ 4x &= 5344000 \\ x &= 1336000 \end{aligned}$$

Thus, the winner received 1 336 000 votes.

400 points: In the diagram below, a circle is inscribed in a large square and a smaller square is inscribed in the circle. If the area of the large square is 36, what is the area of the smaller square?



Since the area of the large square is 36, then the side length of the large square is 6. Therefore, the diameter of the circle must be 6 as well, and so its radius is 3. We can divide the smaller square into 4 identical isosceles triangles by drawing two lines connecting the diagonally opposite corners of the smaller square. The base and height lengths of each of these triangles is equal to the length of the radius of the circle (or 3).

We can determine the area of each of these triangles by using the formula $\frac{1}{2}bh$. The area of the smaller square will be the sum of the areas of the four triangles, or $4 \times \frac{1}{2}bh = 2bh = 2 \times 3 \times 3 = 18$.

500 points: A large block, which has dimensions n by 11 by 10, is made up of a number of unit cubes and one 2 by 1 by 1 block. There are exactly 2362 positions in which the 2 by 1 by 1 block can be placed. What is the value of n?

This question is number 25 from Gauss grade 8 2004. The solution can be found at: $http://www.cemc.uwaterloo.ca/contests/past_contests/2004/2004Gauss8Solution.pdf$

<u>**Mystery:**</u>

100 points: For their project, a group of students needed 30 poles, each 6 units in length. They also needed 20 poles, 8 units in length, and ten poles, 10 units in length. The poles were only available in 16 unit lengths. Assuming no gluing and no loss in cutting, what is the least number of 16 unit length poles needed?
Since we need 10 poles that are 10 units in length, these must be cut from ten 16 unit length poles. The left over 6 units can be used for 10 of the 6 unit lengths poles. The remaining 20, 6 unit length poles can be cut from 10, 16 unit length poles. Then the 20 poles, 8 units in length can be cut from 10 poles 16 units in length. The total number of 16 unit length poles used is 30.

200 points: Harry, Ron, Hermione, and Ginny are trying to fly from Hogwarts castle to the Weasley's house but they only have one broom. The broom can only hold one of Harry and Ron or both (or one) of Hermione and Ginny. What is the minimum number of trips between Hogwarts and the Weasley's house needed to move all four people from Hogwarts to the Weasley's house?
The minimum number of trips is 9.
Note: the solution below is only one solution, there are other possibilities.
Trip 1: Ginny and Hermione travel from Hogwarts to the Weasley's house.
Trip 2: Hermione travels back from the Weasley's house to Hogwarts.
Trip 3: Harry travels from Hogwarts to the Weasley's house.
Trip 4: Ginny travels from the Weasley's house to Hogwarts.
Trip 5: Ginny and Hermione travel from Hogwarts of the Weasley's house.
Trip 6: Hermione travels from the Weasley's house to Hogwarts.
Trip 7: Ron travels from Hogwarts to the Weasley's house.
Trip 8: Ginny travels from the Weasley's house to Hogwarts.
Trip 9: Ginny and Hermione travel from Hogwarts of the Weasley's house.

300 points: Tony and Maria are training for a race by running all the way up then back down a 700m long ski slope. They each run at different but constant speeds. When they run back down the slope, each person runs at double his or her uphill speed. Maria reaches the top first, and immediately starts running back down the hill, meeting Tom 70m from the top. When Maria reaches the bottom of the hill, how far behind is Tom?
This question was number 25 from Gauss grade 8 in 2001. The solution can be found at:
$http : //www.cemc.uwaterloo.ca/contests/past_contests/2001/2001Gauss8Solution.pdf$

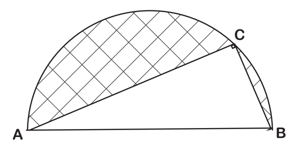
400 points: Joshua, Cassy, and Emily divide a sum of money that they won from the lottery. The sum of money was divided as follows:

- Joshua receives \$775 plus $\frac{1}{5}$ of what remains;
- Cassy receives \$900 plus $\frac{1}{6}$ of what remains; and
- Emily receives the rest, which is \$1000.

How much was the original sum of money?

Since Cassy received $\frac{1}{6}$ of what remains and Emily receives the rest, this means that Emily received $\frac{5}{6}$. So $\frac{1}{6}$ of what remains is \$200. In total, Cassy received \$1100. Then since Joshua receives $\frac{1}{5}$ of what remains, this means that the sum of what Cassy and Emily receive is $\frac{4}{5}$ of the remaining amount. So, $\frac{1}{5} = \frac{2100}{4} = \$525$. In total, Joshua received \$1300. The original sum of money is $\$1000 + \$1100 + \$1300 = \$3400$.

500 points: AB is the diameter of the semicircle in the diagram below. If $AC = 8$, $CB = 6$, and $\angle ACB = 90°$, find the area of the shaded region.



The area of the shaded region is equal to the area of the half circle with radius equal to half of the hypotenuse minus the area of the right triangle. The area of the triangle is $\frac{1}{2}bh = \frac{1}{2}6 \times 8 = 24$. The length of the hypotenuse can be found using Pythagorean theorem.

Hypotenuse=$\sqrt{(CB)^2 + (AC)^2} = \sqrt{(6)^2 + (8)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$.

So the radius of the circle is 5.

Then, the area of the half circle is $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(5)^2 = \frac{25}{2}\pi$.

Thus, the area of the shaded region is $\frac{25}{2}\pi - 24$.

Winner: Team 1! Good job everyone!