UNIVERSITY OF WATERLOO

University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
February 15th, 2012
Ancient Geometric Constructions

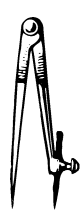
The ancient Greek mathematician **Euclid** is the acknowledged inventor of the area of mathematics known as *Geometry*. These ancient construction techniques of drawing shapes using simply a **compass** and a **straight edge** are used extensively in his book 'Elements', which is regarded as the ultimate geometry reference even today.

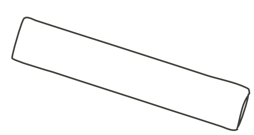
Geometric construction is the *purest* form drawing shapes *without using numbers.* You'll be surprised at what you can do without numbers in mathematics :)

Why did they use no numbers back then? It was because that the number system we have today, such as fractions and decimals, and even the almighty **zero** didn't exist. They had only whole numbers. So there was no way to represent, say $5 \div 2 = 2.5$, back then.

Note: In this lesson, you must NOT use the measurement on the ruler to measure the length of anything!!! The ruler should just be a guide to help you draw straight lines! Remember, ancient Greeks didn't have centimeters and inches!

Instructions for Compass and Straight Edge Constructions

This is a compass: 

This is a straight edge (a ruler with no markings): 

1. You have a compass, which allows you to draw circles and arcs, and you have a straight edge (a ruler in this case, whose markings you must not use) that helps you draw a line between two points.
2. You are allow to make points
3. You are allowed to connect any two points with a line
4. You are allowed to draw a circle centered at point a going through point b
5. You are allowed to mark the point of intersection of any two objects
6. You are allowed no more than the above!

Make sure you know what these are:

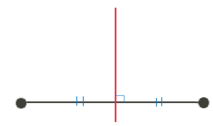
- Perpendicular lines
- Parallel lines
- Bisection point of a line segment
- Equilateral triangle
- Isosceles triangle
- Scalene triangle
- Circle
- Arc
- Diameter
- Radius
- Square
- Sum of angles inside a triangle
- Pentagon
- Hexagon

For instructions to the below example constructions, consult this webpage.
http://www.mathopenref.com/tocs/constructionstoc.html

Let's get started!

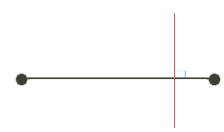
Lines

Example 1: Bisecting a line



Why does this bisect the line?

Example 2: Perpendicular line from any point on a line



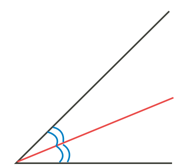
Why is this line perpendicular?

Example 3: Copying a line

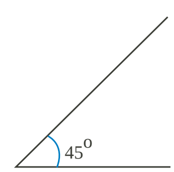
Why are these two lines the same?

Angles

Example 4: Bisecting an angle



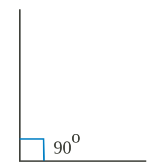
Why does this bisect the angle?



Example 5: Construct a 45° angle

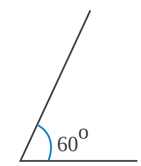
Why is this 45°?

Example 6: Construct a 90° angle



Why is this 90°?

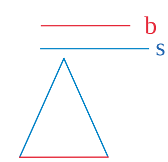
Example 7: Construct a 60° angle



Why is this 60°?

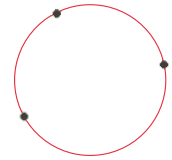
Triangles

Example 8: Isosceles triangle given base and side



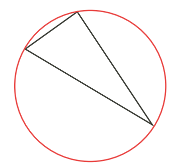
Why is this triangle isosceles?

Circles



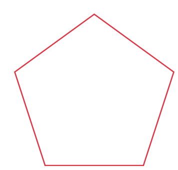
Example 9: Circle given 3 points

Note: 3 points always uniquely define a circle (i.e. there is exactly 1 circle that passes through all 3 points).



Example 10: Circumcircle of a triangle

Shapes

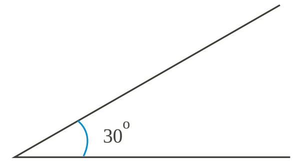


Example 11: The ultimate construction of all — Pentagon

What is the degree of the inside angles of a pentagon? Total degree of a pentagon?

Exercises:

Try to construct these objects following the steps outlined below:



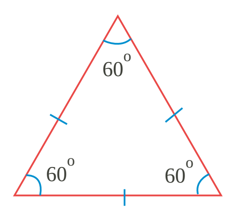
Problem 1. Construct a $30°$ angle using the following 2 methods

Method 1: Circles method

1. Draw 2 points, label them P and Q
2. Draw a line through points P and Q, this is the line PQ
3. Set your compass width to any width as long as it's shorter than the line PQ
4. Using P as the center, draw an arc with the compass width such that it crosses the line PQ
5. Label the intersection point of the arc and the line as S
6. Without changing the compass width, set the center of the compass to the point S
7. Draw an arc centered at S such that this new arc crosses your previous arc
8. Mark the point of intersection of both arcs as R
9. Without changing the compass width, set the center of the compass to the point R
10. Draw an arc centered at R such that this new arc crosses the second arc you drew
11. Mark the point of intersection of the two arcs and label it T
12. Draw a line from P to T
13. Voila! The angle TPQ has degree $30°$

Method 2: Bisect a $60°$ angle

1. Construct a $60°$ angle as did in class.
2. Bisect this angle as did in class.



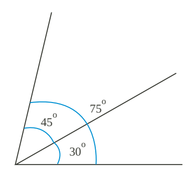
Problem 2. Construct an equilateral triangle

A). Construct the triangle PQR:

1. Choose any desired side length of your equilateral triangle, set the width of your compass to it.
2. Make a point P
3. Center your compass at P, draw a circle centered at P using the width you set your compass to
4. Pick anywhere on the circumference of the circle, call this point Q
5. Without changing the width of your compass, center your compass at Q
6. Draw a circle centered at Q using the width you set to your compass (to check that you are correct, this circle should pass through P)
7. Mark any point of intersection of the two circles, label this point R
8. Connect PQ, QR and RP
9. Voila! The triangle PQR is an equilateral triangle

B). Prove that the triangle PQR you made in part A). is equilateral.

Problem 3. Construct a combination angle $75°$



1. Construct a $45°$ angle as we did in class
2. Choose any of the sides of the $45°$ angle and using this as your base line, construct a $30°$ angle, as did in Problem 1
3. The combined angle of the $45°$ and $30°$ is your $75°$ angle

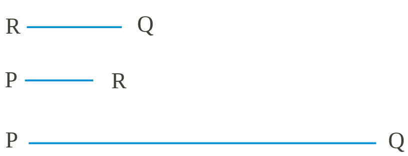
Problem 4. Construct a triangle given 3 sides

A). Construct a triangle

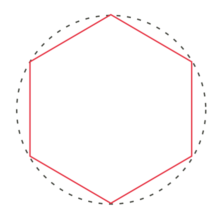
1. Draw 3 lines, label them PQ, PR, and QR, make sure that **the combined length of any 2 lines is longer than the 3rd line!** These 3 lines are the length of the sides of your triangle.
2. Set your compass width to the length of the line PQ
3. Make a point, call it P
4. Draw an arc with radius being the length of PQ, with your compass centered at P
5. Label any point on the arc as Q
6. Set your compass width to the length of the line PR

7. Draw an arc with radius being the length of PR, with your compass centered at P
8. Set your compass width to the length of the line QR
9. Center your compass at QR
10. Draw an arc with radius being the length of QR, with your compass centered at Q. Draw this arc such that it crosses the arc with radius PR that your drew previously
11. Mark the intersection point of the 2 arcs, label this point R
12. Connect points PQR, this is your triangle of the desired given size.

*B). Try to construct a triangle using exactly the lengths given below, are you encountering any problems with the construction? Can you explain why this problem occurs? (Hint: Look at the bolded text in part A).)



Problem 5. Construct a hexagon inside a circle



A). Construct a hexagon

1. Choose any side length of the hexagon you desire, make points A and B
2. Draw a line connecting A and B, the line AB is the side length of your hexagon
3. Set your compass width to the length of AB
4. Place your compass centered at A, draw a circle with radius being the width you set your compass to
5. Do the same for your compass centered at B (To check that the above two steps are correct, the two circles centered at either point should pass through the other point)
6. Mark either of the two intersection points of the two circles, label this point O
7. Without changing the width of your compass, center it at O
8. Draw the circle centered at O (to check that you are correct, this circle should pass through points A and B)
9. Without changing the width of your compass, center it back at A

10. Draw an arc centered at A such that the arc crosses the circle you drew previously (the circle centered at O)

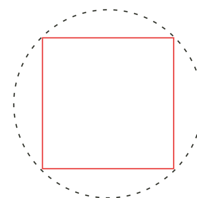
11. Mark the intersection point of the circle and the arc, label this point F

12. Repeat steps 9-11 for the next 4 such points on the circle starting at F, label the new intersection points E, D, C

13. Connect these points, namely the points $ABCDEF$. This is your hexagon!

B). Now draw three lines crossing the diagonals of the hexagon, you should get 6 little triangles. What type of triangles are these and why?

Problem 6. Construct a square inside a circle



A). Construct square $ABCD$

1. Choose any point, label this O

2. Set your compass to any desired width, draw a circle centered at O

3. Without changing the width of your compass, choose any point on this circle, label this point M

4. Connect points O and M

5. Draw a circle of the same width centered at M (To check that you are correct, this circle should pass through O)

6. The two circles will intersect at 2 points, name them P and Q

7. Draw a line connecting P and Q

8. Mark the intersection points of the line PQ and OM, call this point R

9. Center your compass at R, set it to the width of the line segment OR

10. Draw a circle of this width centered at R (To check that you are correct, this circle should pass through O and M)

11. Mark the two intersection points of this circle and the line PQ, label these points S and T

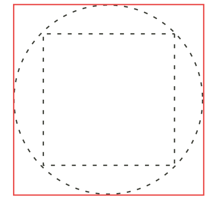
12. Draw a line segment going from O to S such that this line is long enough to pass through the first circle you drew (the big circle centered at O)

13. Do the above step for a line segment going through O and T

14. These two diagonal lines should intersect with the big circle at 4 points. Label these 4 points $ABCD$

15. Connect these 4 points. There is your square!

B). Construct square $EFGH$



1. Keep the large circle you had from part A). (the one centered at O) as well as your square $ABCD$ (you can erase the other things if they are distracting)

2. Draw a **perpendicular bisector** of the edge AB, as did in class, make it long enough so that it passes its intersection with the circle.

3. Same as step 2). Draw this perpendicular bisector for edge BC

4. These two perpendicular bisectors should intersect the large circle at 4 points. Label these points U, V, W, X

5. Draw **perpendicular (note: not perpendicular bisector, just perpendicular)** lines to the lines UW and VX at the 4 points: U, V, W, and X (There should be 4 perpendicular lines draw after this step). Draw these perpendicular lines long enough so that they intersect.

6. Label the 4 intersections of these perpendicular lines $E, F, G, andH$

7. This is your bigger square $EFGH$! (To check that you are correct, your circle centered at O should be inside this square, touching this square at the points U, V, W, and X)

*C). Prove that square $EFGH$ has twice the area as square $ABCD$.