University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
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Algebraic Approaches

Algebra is a very broad term in mathematics. It refers to any part of mathematics where letters and other symbols are used to represent numbers and quantities in formulae and equations. Look at that!

Today we will explore some problem solving where we must exploit algebraic techniques. The biggest difference between algebra problems and geometry problems in problem solving is that algebra problems generally have techniques to get you started when approaching a problem. On the other hand, geometric problems, especially classical Euclidean Geometry problems can be very flexible and varied, having you knowing only the basic facts and must derive the intermediate steps on your own, A.K.A. Harder! Most of these problems today look hard to begin with, but once you learn the approach, they become routinely, sort of.

We'll start off with a miscellaneous problem:

$$\begin{array}{r} \dagger\star 7 \\ -\ 169 \\ \hline 258 \end{array}$$

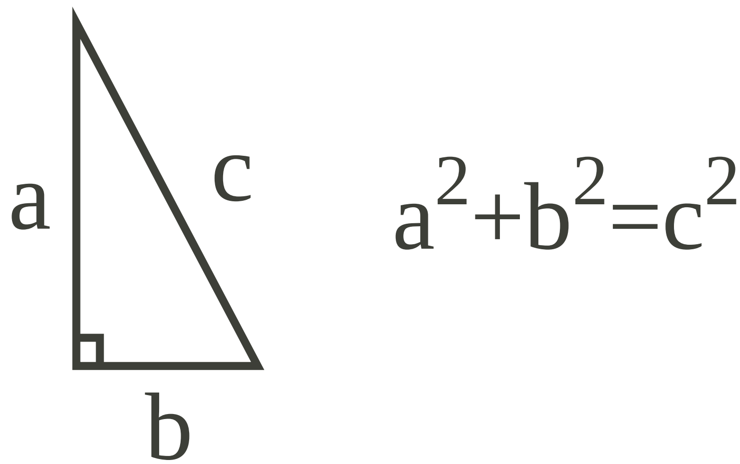
What is †, what is ⋆?

The Pythagorean Theorem

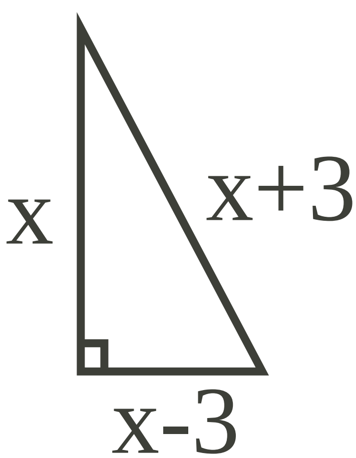
This is not strictly algebra, but it's an interesting cross reference between equation solving and geometry.

On any **right-angled triangle**, we say that the **hypotenuse** is the edge facing the 90° corner. (i.e. the longest edge).

Then the relationship of the length between the 3 edges on a right-angled triangle is modeled as follows:



Problem 1:
Determine the value of x:



$$\begin{aligned} x^2 + (x-3)^2 &= (x+3)^2 \\ 2x^2 - 6x + 9 &= x^2 + 6x + 9 \\ x^2 - 12x &= 0 \\ x(x-12) &= 0 \end{aligned}$$

$x = 0$ or $x = 12$
$x = 12$ is the only solution, since length cannot be 0.

Exponents, Associativity, Distributivity

1). $2^{a+b} = 2^a \times 2^b$

2). $6a + 18b = 6(a + 3b)$

3). $(2^a)^b = 2^{ab}$

4). $2^2(3^a + 5^b) = 4 \times 3^a + 4 \times 5^b$

5). $(ab)^n = a^n b^n$

Problem 2:
What is the largest integer n such that $5(n^{2012}) < 5^{4025}$?

$$\begin{aligned} 5(25^{2012}) &= 5(5^2)^{2012} \\ &= 5(5)^{4024} \\ &= 5^1 \times 5^{4024} \\ &= 5^{4025} \end{aligned}$$

So if $n = 25$, we have $5(n^{2012}) = 5^{4025}$. Then for $5(n^{2012}) < 5^{4025}$, the greatest integer n must be 24.

Problem 3:
Determine all possibles values (x, y) such that $2^{x+1} + 3^y = 3^{y+2} - 2^x$

$$\begin{aligned} 2^{x+1} + 3^y &= 3^{y+2} - 2^x \\ 2^{x+1} + 2^x &= 3^{y+2} - 3^y \\ 2^x(2+1) &= 3^y(9-1) \\ \frac{2^x}{8} &= \frac{3^y}{3} \\ \frac{2^x}{2^3} &= \frac{3^y}{3^1} \\ 2^{x-3} &= 3^{y-1} \end{aligned}$$

The only time that a power of 2 is equal to a power of 3 is $2^0 = 1 = 3^0$.
That is, $x - 3 = 0$ and $y - 1 = 0$. So $x = 3$ and $y = 1$ is the only solution for x, y.

The Pigeonhole Principle

In its strict sense, the Pigeonhole Principle is a combinatorial result. The idea is very simple.

If I have 9 pigeons and only 8 holes to hold them, then at least one of the holes must have more than 1 pigeon, right?

i.e. The Pigeonhole Principle states that given n items and p holes to put them in, where $n > p$. At least one of the p holes has to have more than 1 item in it.

Problem 4:
How many people do you need in a party at minimum to have 2 people born in the same month?

13

Problem 5:
25 students each earn a grade of A, B, or C, the most frequently occurring letter grade be the grade for at least __ students?

9

Averages are Fun

The average value of n numbers is the sum of the numbers divided by how many numbers there are:

$$\textbf{Average} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n}$$

Where each x_i is the value of that number.

Problem 6:

If the average of 3 numbers is 15, what is their sum?

$$\text{sum} = 3 \times 15 = 45$$

Problem 7:

My average over 3 rounds of Rubix Cube solving is 5 minutes and 21 seconds. (Yes I know it is very slow!) But I did a very good 4th round and lowered my average time over 4 rounds to 4 minutes and 16 seconds. (Yeah! That's some pretty good improvement!). How fast did I solve the Rubix cube in my 4th round?

Let's convert the minutes to seconds for easy calculation.

5 min and 21 sec = 321 seconds; 4 min and 16 sec = 256 seconds.

Let x be the number of seconds I took on my 4th round to solve the Rubix Cube.

$$\begin{aligned} 321 \times 3 + x &= 256 \times 4 \\ 963 + x &= 1024 \\ x &= 61 \end{aligned}$$

It took me 61 seconds to solve the Rubix Cube on the 4th round. i.e. 1 minute and 1 second.

A Different Way to Write Integers

Let $b = 7$, and $a = 4$. What is $10b + a$?

$10b + a = 10(7) + 4 = 74$

How can you write a 3 digit integer this way?

$100c + 10b + a$, where $a, b, c \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, c \neq 0$.

Problem 8:
A 2-digit number minus 54 equals the 2-digit number but with the digits reversed. Find all possible such 2-digit numbers.

We may rewrite this question as: $10b + a$-54=$10a + b$

$$\begin{aligned} 10b + a - 54 &= 10a + b \\ 9b - 9a &= 54 \\ 9(b - a) &= 54 \\ (b - a) &= \frac{54}{9} \\ (b - a) &= 6 \end{aligned}$$

Notice that none of b or a can be 0, since that would make either $10b+a$ or $10a+b$ a single digit number.

We are looking for pairs of single digits a, b such that $b - a = 6$. The possible such pairs are $(a = 1, b = 7), (a = 2, b = 8)$, and $(a = 3, b = 9)$

Together We Are Strong!

Construction problems usually refer to getting some set amount of work done by a working unit over some amount of time.

Here is what to remember:

Amount per unit × # of units = Total amount

Here is what you should NEVER do:

Moooooo! If cow-1 is eating the grass, the grass will last cow-1 4 hours. If cow-2 is eating the same patch of grass, the grass will last cow-2 2 hours. If cow-1 and cow-2 both eat the grass, then the grass will last them 3 hours, because $\frac{4+2}{2} = 3$ since cow-1 will eat some and cow-2 will eat some but neither will eat all, so the grass will last their average.

This approach is INCORRECT and there is no way to reason behind this approach other than your sheer intuition telling you that you should take their average.

Problem 9:

The cow eating grass problem on the previous scenario with the question being: if cow-1 and cow-2 both eat the grass together, how long will the grass last them?

Let the total amount of grass be n.

Cow-1 can eat $n \div 4$ amount of grass per hour; Cow-2 can eat $n \div 2$ grass per hour.

Then cow-1 and cow-2 combined can eat $\frac{n}{4} + \frac{n}{2} = \frac{3n}{4}$ amount of grass per hour.

Then the patch of grass n will last cow-1 and cow-2, together,

$$n \div \frac{3n}{4} = n \times \frac{4}{3n} = \frac{4}{3} \text{ hours}$$

i.e. 1 hour and 20 minutes.

Problem 10:

Snow shoveling, yay! It's a snow day and Kevin doesn't have to go to school, but he has chores to do: driveway cleaning! To shovel Kevin's driveway, Kevin takes 12 hours alone. Clare would take 6 hrs to shovel Kevin's driveway. Meanwhile, John would take 4 hours to shovel the same driveway. Since John and Clare are such good friends of Kevin, they decide to help Kevin shovel his driveway together. How long would it take the 3 of them to shovel Kevin's driveway?

2 hours

We'll end with a miscellaneous problem:

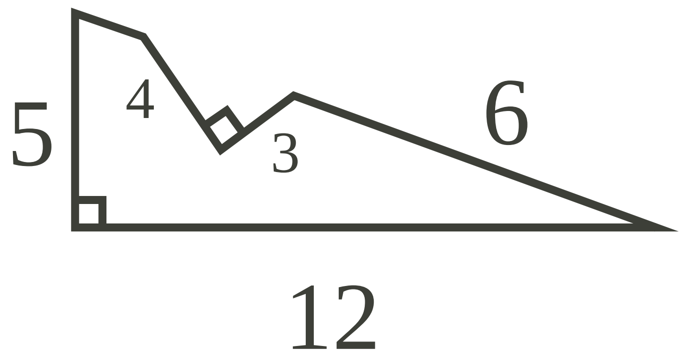
The digits 1, 2, 3, 4, 5 and 6 are each used once to compose a six digit number $abcdef$ such that the three digit number abc is divisible by 4, bcd is divisible by 5, cde is divisible by 3 and def is divisible by 11. What is this number $abcdef$?

$$abcdef = 324561$$

(Hint: Look for where to start on the problem, which number is easy to start tackling down. Try to remember your divisibility rules, and in case you didn't know, a number is divisible by 11 if and only if the sum of its odd-positioned digit minus the sum of its even-positioned digit is divisible by 11. For example, 231 is divisible by 11 because $2 + 1 - 3 = 0$ and 0 is divisible by 11; likewise 3927 is divisible by 11 because $7 + 9 - (2 + 3) = 11$ which is divisible by 11.)

Exercises:

1. Determine the area and perimeter of the following figure, assuming the two pieces of the hypotenuse of the larger "triangle" lie on the same line.



2. What is the value of
$$\frac{2^{2012} - 2^{2011}}{2^{2012} + 2^{2011}}$$

3. Find the largest integer n such that $n^{200} < 3^{500}$?

4. Arrange these from smallest to largest: $3^{666}, 4^{555}, 5^{444}, 6^{333}, 7^{222}$.

5. My average in my math class was 75. I did a really good unit test last week and got a 84% on it. Now my new average is 78. I'm so close to an 80, which will make me an honour student. What must I get on my next math unit test to become an honour student?

6. Say for example, all digits of a number are even. Then we can call it "completely even". We may call 264 completely even, but 254 is not completely even. What is the sum of all 3-digit completely even numbers?

7. Arthur is driving to David's house, intending to arrive at a certain time. If he drives at 60 km/h, he will arrive 5 minutes late. If he drives at 90 km/h, he will arrive 5 minutes early. What speed does he have to drive at to arrive exactly on time? (75 km/h is INCORRECT)

8. The new iPads have just arrived (this is fictitious)! Employee Lisa at the Conestoga Apple Store can stock and decorate the iPad shelf with them in 8 hours. Both Lisa and Jeff stocked and decorated the iPad shelf. It took them 5 hours together. If Jeff were to decorate the shelf on his own, how long would it take?

9. *Optional: (If you are familiar with factoring quadratics, try this) Determine all possible values of x.

$$2^{2x} - 3(2^x) - 4 = 0$$

10. *How many people do you need at a party at minimum such that there is guaranteed to have 3 people that are either mutual friends or mutual strangers? (A, B, C are mutual friends if each one of them is a friend with the other two; A, B, C are mutual strangers if neither of them knows any of the other two.)