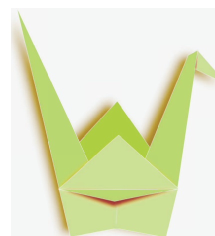
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
February 29th, 2012
Spatial Visualization and Origami

Everybody loves origami! Origami is a traditional Japanese craft. "Origami" literally means "Folding Paper" with "Oru" in Japanese meaning to fold, and "Kami" meaning paper. It started in 17th century AD in Japan and was popularized in the west in mid 1900's. It has since then evolved into a modern art form. Today, designers around the world work with this exquisite art form to make all kinds of wonderous things! The goal of this art is to transform a flat sheet of material into a finished sculpture through folding and sculpting techniques. Cutting and glueing are not part of strict origami. Origami with cutting is a modified form called *kirigami*



Today's lesson, we will explore the mathematical wonders of this sophisticated craft as well as focus on how this art form has helped in our capability to visualize geometric transformations in space.

Definitions:

We are interested in the particular transformation called *reflections*:

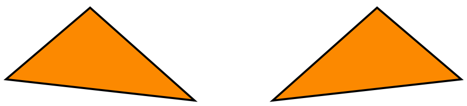
A **reflection** is a transformation where an object is symmetrically mapped to the other side of the *line of symmetry*.

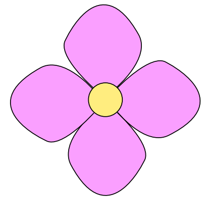
A **line of symmetry** is also known as the *mirror line* where the object look the same on both sides of the "mirror".

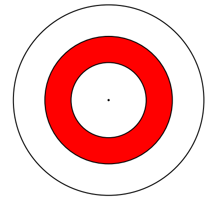
Creases on a sheet of folded paper are the lines that you folded along after you open up your folded origami.

Examples:

Identify the number of lines of symmetry in the following figure.

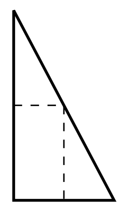






Identify Geometry Properties using Origami

Let's see some geometry first. Some properties of a geometric shape and formulas we use in calculating area of a shape can be easily explained.

1. Dividing the hypotenuse in half on a right angled triangle. 

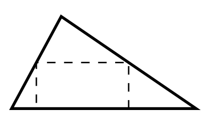
   (a) Fold along the hypotenuse such that the upper tip of your triangle touches the bottom of the hypotenuse, open it up, this point is half way of the hypotenuse.

   (b) Fold along the bottom edge of your triangle such that this is also bisected.

   (c) Fold along the side edge of your triangle such that this is also bisected.

   How do the resulting area of the smaller triangles compare to the big triangle?

   What relationship do you notice amongst the small triangles?

2. Rectangle half the height of the triangle. 

   (a) Grab any triangle, choose the longest side to be your base (This is to eliminate not seeing the rectangle on an obtuse triangle, for an acute triangle, you may choose any side to be your base).

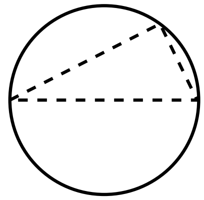
   (b) Fold the top corner down so that it touches the base and the crease created is parallel to the base.

   (c) At the point of intersection of the crease from folding down the top corner and the right edge of the triangle, fold in the right corner such that it makes a crease perpendicular to the base.

   (d) Repeat step 2). for the left corner.

   (e) To check that you are correct, you should get a rectangle formed by the 3 creases and the base of your triangle.

For the rectangle enclosed by the creases on paper, what is the area of this rectangle compared to the big triangle? Can you explain why?



3. Triangle inscribed in a half circle.

   (a) Fold the circle in half, the crease is the diameter of the circle.

   (b) On your half circle, fold any part such that you get a sharp corner with a chord crossing a part of an arc of the semi-circle.

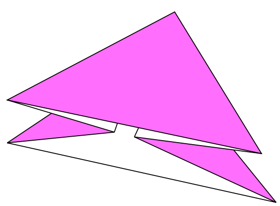
   (c) Fold down the piece of the arc so you get a triangle.

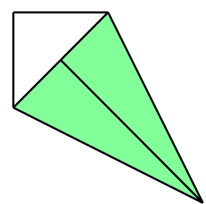
   What do you notice about the angle opposite the diameter? Can you explain why?

Colourability of Origami

A **flat fold** is a fold such that the resulting object can lie flatly on a surface (i.e. the resulting object is 2D).

Let's do the following investigation on flat folds.
We'll fold a water-balloon base as well as a kite base to illustrate the colourability of flat folds.

Water-balloon base: 

Kite base: 

Example:

1. Fold a water-balloon based figure (simple frog) as instructed and open up your fold.

   Now the task is to grab your colour pencils and try to colour this using different colours such that no two adjacent pieces are the same colour:

   Can you colour this using 4 different colours?
   Can you colour this using 3 different colours?
   Can you colour this using 2 different colours?

2. Fold a kite based figure (swan) as instructed and open up your fold.

   Can you colour this using 2 different colours?

Theorem: Any opened up origami paper that has the crease of a flat-fold can be coloured using just __ colours.

Consequence of the theorem: If you can colour it with 2 colours, then it cannot contain __ pieces that are adjacent to each other in its opened up crease. **Then we may draw a conclusion that all flat-fold origami has no ____________________________________.**

Creases and Symmetry in Origami

A lot of origami folds are symmetrical on both sides without having you fold them twice.This is especially true when creating origami animals.

For example, the famous paper crane is symmetrical.

Why is this?

This is because what is done on one side is reflected along the lines of symmetry as you fold your paper.

A **inner most corner** is the corner(s) after a series of fold where you see the least number of layers of sheet paper.

A **outer most corner** is the corner(s) after a series of fold where you see the most number of layers of sheet paper.

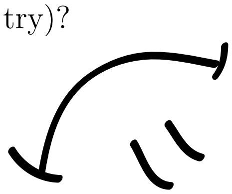
Examples:

1. How does the initial face become the transformed face(describe the transformations and the lines of symmetry)?

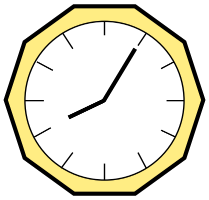

Initial face


Transformed face

2. How does the initial face become this (describe the transformations and the lines of symmetry)?



3. You are sitting facing a mirror and see this in the mirror. What time is it actually?



Good!! You are ready for more challenge.

Definition:

A **kirigami** is origami with cutting! In the strict definition of origami, cutting is not involved. But in kirigami we may fold and cut, for example, paper snowflakes!!! Yay snowflakes!

The **smallest component** is the smallest portion of a symmetrical figure such that it cannot be generated by reflecting a smaller component against a line of symmetry. (i.e. It is asymmetric)The entire figure is generated by repeatedly reflecting this component against multiple lines of symmetry.

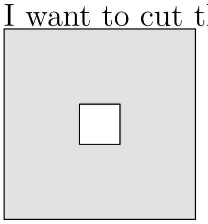
Being able to visualize the crease and cuts after a series of folding and cutting is a critical part of developing one's spatial sense. We'll illustrate some simple techniques using lines of symmetry of origami and kirigami.

Strategy:

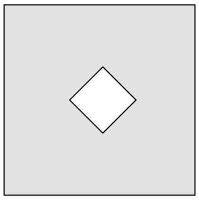
1. Identify the smallest component of your entire structure that is independent on its own. (i.e. the smallest component that cannot be constructed using reflection on some line from a smaller component)
2. Remember, every time you fold your paper in half, you save twice the amount of cutting. (likewise, if you fold your paper in thirds, you save three times the amount of cutting, etc.)
3. In general, it is *easier* to FOLD identical pieces than CUT identical pieces.
4. Whenever you cut after a fold, this cut becomes symmetrical on the other side of your fold line.

Examples:

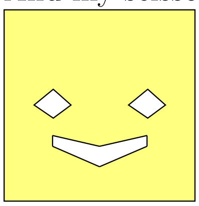
1. I want to cut this in one stroke of scissor. How do I fold before I cut?



2. I want to cut this in one stroke of scissor. How do I fold before I cut?



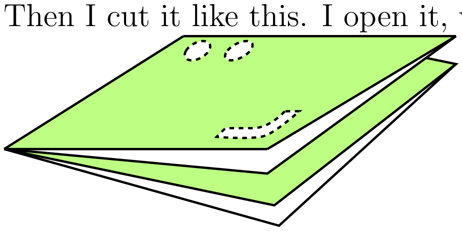
3. I want to cut this such that the eyes are symmetrical along the middle and so is the mouth. And my scissor cannot dig into the paper. How do I fold before I cut?



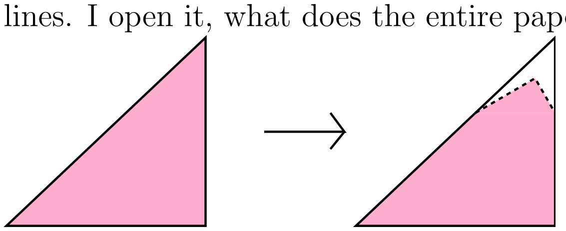
4. I fold my paper in half, then in half again, creating a square a quarter of the original size. Then I fold the corner where I can see the sheets (outermost corner) towards the innermost corner, creating a triangle $\frac{1}{8}$ the original size. I open it, what does the crease look like?

5. I fold my paper in half, then in half again, creating a triangle a quarter of the original size. Then I fold the innermost corner towards the bottom of the triangle (outermost edge) such that it just touches the edge, creating a trapezoid. Then I open it, what is the area of the square enclosed by the crease mark with respect to the original square piece of paper?

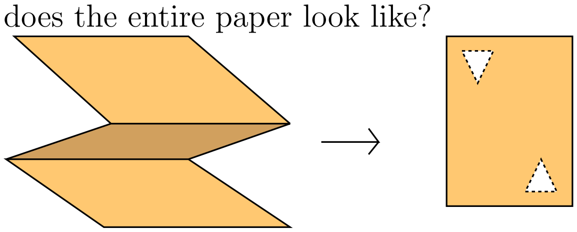
6. I fold my paper in half, then in half again, creating a square a quarter of the original size. Then I cut it like this. I open it, what does the entire paper look like?



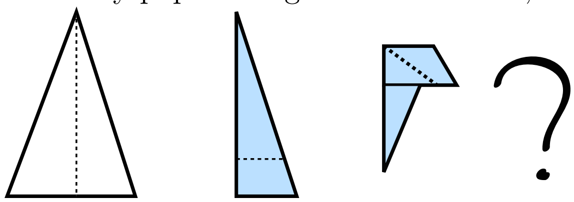
7. I fold my paper in half, then in half again, creating a square a quarter of the original size. Then I fold this square in half again with the crease dividing the innermost corner, creating a triangle a quarter of the original size. Then I cut the inner most corner along the dotted lines. I open it, what does the entire paper look like?



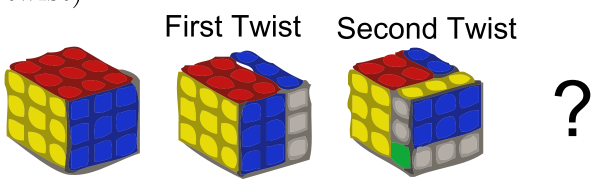
8. I fold my paper in three, once over and once under. Then I cut it like this. I open it, what does the entire paper look like?



9. I fold my paper along the dotted lines, what does the final figure look like?



10. This is Robin's Rubix cube. She twisted it 3 times clock-wise on on the different faces. This is after the 1st twist. This is after the 2nd twist. What does it look like after the 3rd twist? (note, you don't need to know anymore colours of the cube to draw the cube after the 3rd twist)



Using Origami in the Real World

To wrap up the month and celebrate this very special day in 4 years, we'll do some real origami!

Are origami useful in real life, you may ask? Well here are some:

1. Box
2. Tea Plate
3. Tall Party Hat
4. Wallet

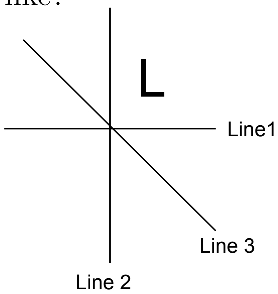
   And last but not least... the most useful origami you'll ever learn...

5. Paper cup

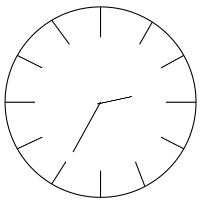
   Which of the above are 2-colourable?

Exercises:

1. The ”L” is reflected in line 1, then in line 2, then in line 3. What does the resulting ”L” look like?

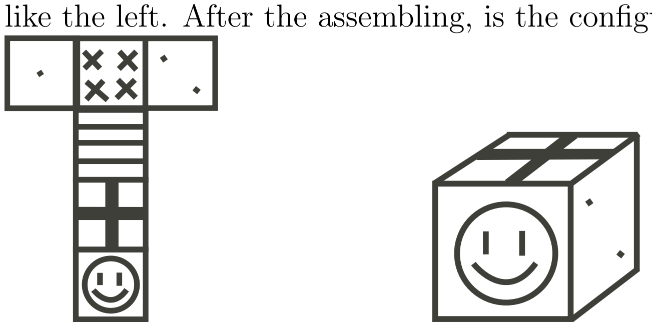


2. This is the time in the mirror, what time is it actually?

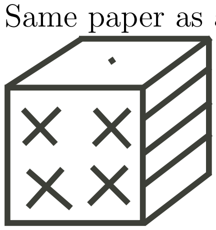


3. Length of arc enclosed by any 2 chords on a circle: use a circle sheet of paper, fold it in half, then fold anywhere to create a chord that encloses a portion of the circle's arc. Unfold the paper. Now you have 2 chords. What can you say about the length of any 2 arcs on a circle enclosed by 2 chords of the same length?

4. Grace bought an interesting piece of paper to assemble a cube. The unassembled version looks like the left. After the assembling, is the configuration on the right possible?

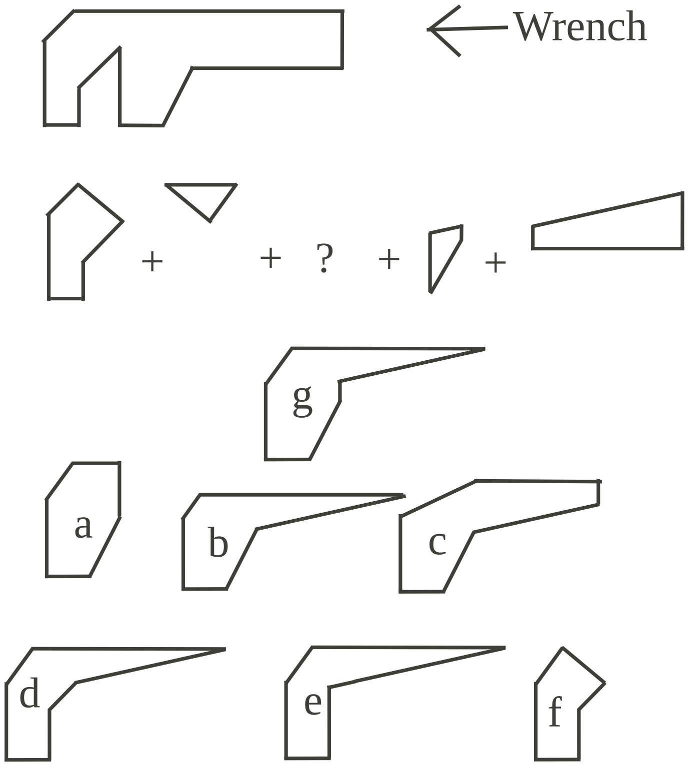


5. Same paper as above. After assembling, is this configuration possible?



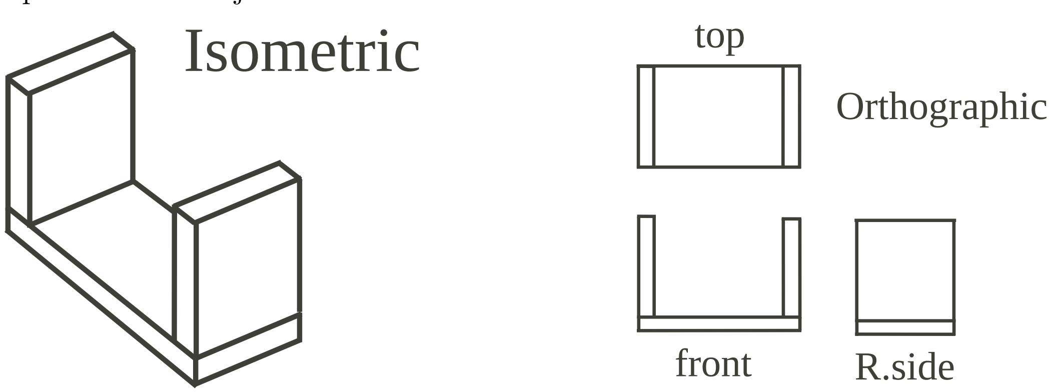
6. I take my square Double Bubble Gum wrapper (presumably with no crease before I fold) and fold it 4 times, creating an isosceles triangle each time. I want to create a rhombus in the middle of the unfolded paper so I can spit out my gum in there and wrap it. Does this fold technique create the rhombus I want?

7. This is a wrench made out of cardboard! Which piece am I missing to assemble the full wrench?

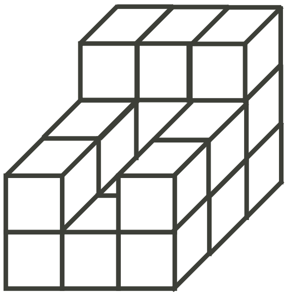


A basic technical drawing of an object usually consists of two forms: **Othographic** and **Isometric**. Orthographic drawing of the object shows the object exactly as how one would see it from front, top and typically, right side view. Isometric drawing of the object is the object rendered in 3D where most of the front, top and right side are visible.

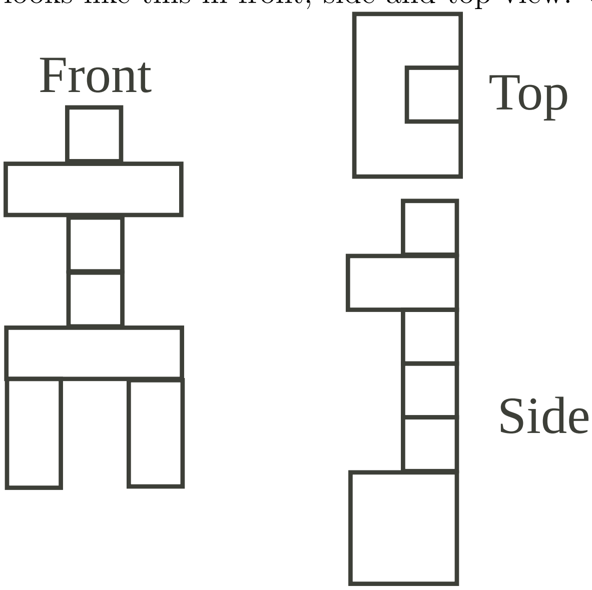
For example, the below are the isometric and orthographic drawings of the same "desk flipped upside down" object:



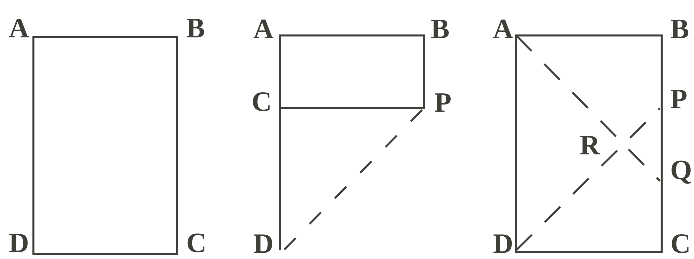
8. This is my very comfortable chair made out of wooden blocks, in 3D! What are the front, side, and top view of this chair?



9. The Inukshuk is the symbolic rock of our gorgeous northern province Nunavut where you can gaze upon the aurora borealis in the crystal clear night sky. There, the Inuit people carry on their beautiful traditions and customs everyday. I made it out of wooden cube blocks and it looks like this in front, side and top view. Can you render this for me in 3D?

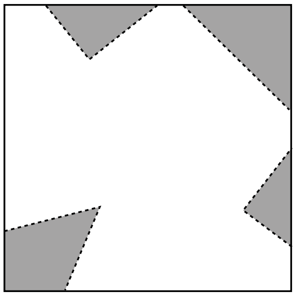


10. A rectangular piece of paper $ABCD$ is folded so that edge CD lies along edge AD, making a crease DP. It is unfolded, and then folded again so that edge AB lies along edge AD, making a second crease AQ. The two creases meet at R, forming triangles PQR and ADR as shown. If $AB = 5$ cm and $AD = 8cm$, what is the area of the quadrilateral $DRQC$ in cm^2.

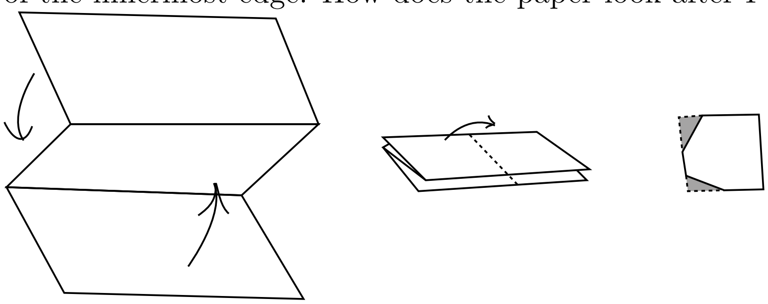


Kirigami

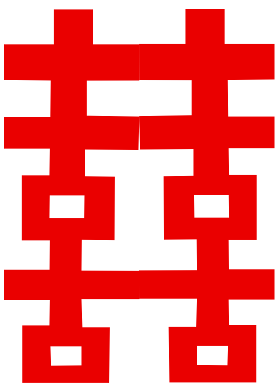
11. I fold my square paper in half twice such that I end up with a smaller square $\frac{1}{4}$ the original size. Then I cut out the grayed region, where the sharp *quadrilateral* (a 4 sided shape) is on the inner most corner of the fold. Unfold the paper, what does my unfolded paper look like?



12. I have a sheet of paper folded into 3 equal pieces, once over and once under. Then I fold in half again along the dotted line. Now I cut off the gray area, with the cut off area being part of the innermost edge. How does the paper look after I unfold?



13. The Chinese celebrates New Years using this special Chinese Character. It symbolizes happiness and prosperity. Identify the number of times we fold a sheet of paper to get this character into its smallest component (i.e. no more lines of symmetry). Can you find a way to cut this character from your folded sheet of paper?



14. Trisecting the angle: Everybody loves cutting snowflakes!! They are easy to make and are pretty, perfect for any festivals especially Christmas! But do you find that cutting 4 or 8 cornered snowflakes are easier than 6 cornered snowflakes, even though snowflakes are actually 6 cornered? This is because it's much easier to continually fold the paper in half each time. But it's much harder to trisect an angle. Try to cut a 6 cornered snowflake by folding the paper such that you get a regular hexagon after you unfold (this involves cutting out some corners after a series of folding). Your should still be cutting on the smallest component.(i.e. you only need to cut a pattern once to get it to appear on the 6 sides of your hexagon)