University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
March 28, 2012
Gauss Contest Preparation

Solutions:

1. Evaluate each of the following:
   a) $5-[2-(1-3)]$ b) $12+6\div 3\times 2-1$ c) $5\times 10^5+5\times 10^3+5\times 10^2+5$
   d) $0.8-0.07$ e) $\frac{0.02}{0.8}$ f) $\frac{x-y}{x+y}$ when x=20 and y=12

   a) $5-[2-(1-3)]=5-[2-(-2)]=5-[4]=1$
   b) $12+6\div 3\times 2-1=12+2\times 2-1=12+4-1=15$
   c) $5\times 10^5+5\times 10^3+5\times 10^2+5=500000+5000+500+5=505505$
   d) $0.8-0.07=0.73$
   e) $\frac{0.02}{0.8}=0.025$ f) $\frac{x-y}{x+y}=\frac{20-12}{20+12}=\frac{8}{32}=\frac{1}{4}$

2. Arrange the numbers $\frac{4}{5}, 81\%, 0.801, \frac{82}{100}, 0.811$ from smallest to largest.
   Order: $\frac{4}{5}, 0.801, 81\%, 0.811, \frac{82}{100}$

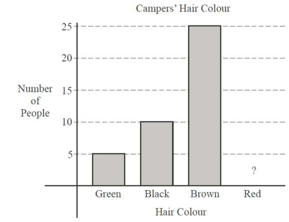
3. Which of the following is the largest?
   a) $\frac{4}{2-\frac{1}{4}}$ b) $\frac{4}{2+\frac{1}{4}}$ c) $\frac{4}{2-\frac{1}{3}}$ d) $\frac{4}{2+\frac{1}{3}}$ e) $\frac{4}{2-\frac{1}{2}}$

   $\frac{4}{2-\frac{1}{4}}=\frac{16}{7}\simeq 2.28$ $\frac{4}{2+\frac{1}{4}}=\frac{16}{9}\simeq 1.78$ $\frac{4}{2-\frac{1}{3}}=\frac{12}{5}\simeq 2.4$ $\frac{4}{2+\frac{1}{3}}=\frac{12}{7}\simeq 1.71$
   $\frac{4}{2-\frac{1}{2}}=\frac{8}{3}\simeq 2.67$
   The answer is E.

4. If P=3 and Q=6, which of the following expressions is **not** equal to an integer?
   a) $P+Q$ b) $P\times Q$ c) $Q\times P$ d) $\frac{P}{Q}$ e) $\frac{Q}{P}$

   $\frac{P}{Q}=\frac{3}{6}=\frac{1}{2}$ The answer is D.

5. The bar graph shows the hair colours of the campers at Camp Gauss. The bar corresponding to redheads has been accidentally re-moved. If 50% of the campers have brown hair, how many of the campers have red hair?



   Looking at the graph, 25 campers have brown hair which represents 50% of the campers. Thus, there are $25\times 2=50$ campers in total. From the graph we can see that 10 campers have black hair and 5 campers have green hair. So, the number of campers with red hair is $50-25-10-5=10$ campers have red hair.

6. Bob has a box that contains 3 red marbles, 6 blue marbles, 6 pink marbles, and 2 purple marbles. Bob then adds a number of black marbles and tells Jack that if he now draws a marble at random, the probability of it being red or blue is $\frac{3}{7}$. How many black marbles did Bob add?

   Let the number of black marbles in the box be equal to b. There are 3 red marbles and 6 blue marbles and 17 marbles in total. We know that the probability of selecting a red or blue marble after the black marbles are added is $\frac{3}{7}$. So,

$$\begin{aligned} \frac{3+6}{17+b} &= \frac{3}{7} \\ \frac{9}{17+b} &= \frac{3}{7} \\ 51+3b &= 63 \\ 3b &= 12 \\ b &= 4 \end{aligned}$$

7. A perfect number is an integer that is equal to the sum of all of its positive divisors, except itself. For example, 28 is a perfect number because $28 = 1+2+4+7+14$. Which of the following is a perfect number?

   a) 10 b) 13 c) 6 d) 8 e) 9

   The positive divisors of 6 are 1,2,3, and 6. $1+2+3=6$. So, the answer is C.

8. When Francis wrote the Gauss Contest, he averaged 1 minute per question on the 10 questions in part A, 2 minutes per question on the 10 questions in Part B, and 6 minutes on the 5 questions in part C. What is the average time that he spent on each question in the entire contest?

   The total time that Francis spent writing the Gauss contest was $(10 \times 1) + (10 \times 2) + (5 \times 6) = 10+20+30 = 60$. Since there are 25 questions in total, the average time spent on each question was $\frac{60}{25} = \frac{12}{5}$.

9. A 250 m train travels through a 2 km tunnel at 27 km/h. How long is it from the time the front of the train enters the tunnel, to when it completely leaves it?

   Let the time that the front end of the train enters the tunnel be $t = 0$. In order for the train to completely leave the tunnel, the train must travel 2 km + 250 m = 2250m=2.250 km. Since $distance = velocity \times time$, $time = \frac{distance}{velocity} = \frac{2.250}{27} = 0.0833\text{hours} = 5\text{minutes}$

10. A palindrome is a number which remains the same when its digits are written in reverse order. For example, 121 is a palindrome. A cars odometer reads 15951 km. What is the least number of kilometers required for the next palindrome to appear?

    The next closest palindrome number will be 16061. $16061 - 15951 = 110$ km.

11. If $N = 2^5 \times 3^2 \times 7 \times \square$ and 100 divides evenly into N, which of the following numbers could be placed in the box?

    a) 5 b) 20 c) 75 d) 36 e) 120

    In order for N to be divisible by 100, N must contain two factors of 2 and 2 factors of 5 since $100 = 2 \times 2 \times 5 \times 5$. N already contains enough factors of 2, so the number that replaces the $\square$ must contain at least two factors of 5. The only number of the five choices that contains 2 factors of 5 is 75, since $75 = 5 \times 5 \times 3$. The answer is C.

12. What is the value of $1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + ... + 45 + 46 - 47 - 48 + 49$?
$1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + ... + 45 + 46 - 47 - 48 + 49$
$= 1 + (2 - 3 - 4 + 5) + (6 - 7 - 8 + 9) + ... + (42 - 43 - 44 + 45) + (46 - 47 - 48 + 49)$
$= 1 + 0 + 0 + ... + 0 + 0$
$= 1$

13. A rectangular sign that has dimensions 9 m by 16 m has a square advertisement painted on it. The border around the square is required to be at least 1.5 m wide. What is the area of the largest square advertisement that can be painted on the sign?

Since the border around the square must be at least 1.5 m, The dimensions of the interior rectangle is 6 m by 13 m. So, the largest square advertisement that can fit onto the sign is 6 m by 6 m.

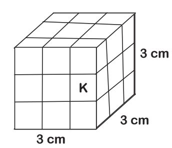
14. The difference between the radii of two circles is 10 cm. What is the difference, in cm, between their circumferences?

Let r_1 and r_2 represent the radii of the two circles. Then, $r_1 - r_2 = 10$ m. The difference in their circumferences is $2\pi(r_1 - r_2) = 2\pi(10) = 20\pi$.

15. In a basketball shooting competition, each competitor shoots ten balls which are numbered 1 to 10. The number of points earned for each successful shot is equal to the number on the ball. If a competitor misses exactly two shots, which of the following scores is not possible?
a) 52 b) 44 c) 41 d) 38 e) 35

If the competitor scores all 10 balls, the total score would be 55. If the competitor misses the first two shots, the score would be 52. If the competitor misses the last two shots, the score would be 36. Of the possible scores listed, the score that is not possible is 35.

16. The figure shown is made up of 27 identical cubes. The cube marked K is removed. What is the effect that this has on the total surface area of the figure?

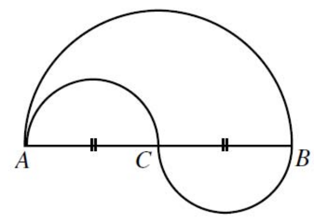


By removing the cube marked by K, the original surface area is decreased by 2 cm^2 since two of the original faces will be removed. When the cube is removed, four inner faces will be exposed and become part of the surface area. So, the net increase in surface area is 2cm^2.

17. Peter is given three *positive* integers and is told to add the first two, and then multiply by the third. Instead, he multiplies the first two and adds the third to that result. Surprisingly, he still gets the correct answer of 14. How many different values could the first number have been?

Let x, y, z represent the three numbers. Then, $(x + y) \times z = 14$ and $(x \times y) + z = 14$. From the first equation, z must be a factor of 14. The factors of 14 are 1, 2, 7, and 14, so z must be one of these values.
If $z = 1$, then $(x + y) = 14$ and $(x \times y) = 13$. This is only possible when $x = 1$ and $y = 13$, or $x = 13$ and $y = 1$.
If $z = 2$, then $(x + y) = 7$ and $(x \times y) = 12$. This is only possible when $x = 3$ and $y = 4$, or $x = 4$ and $y = 3$.
If $z = 7$, then $(x + y) = 2$ and $(x \times y) = 7$. In this case there are no possible values for x and y.
If $z = 14$, then $(x + y) = 1$ and $(x \times y) = 0$. In this case there are no possible values for x and y.
Thus, there are 4 possible values for x.

18. In the diagram, $AC = CB = 10$ m, where AC and CB are each the diameter of the small equal semi-circles. The diameter of the larger semi-circle is AB. In traveling from A to B, it is possible to take one of two paths. One path goes along the semi-circular arc from A to B. A second path goes along the semi-circular arc from A to C and then along the semi-circular arc from C to B. What is the difference in the lengths of these two paths?



For the semi-circular arc from A to B, the length is equal to half of the circumference of the circle with radius 10. The distace would be $\frac{1}{2} \times 2 \times \pi \times 10 = 10\pi$. For the semi-circular arc from A to C, the length is equal to half of the circumference of the circle with radius 5. This is also true for the semi-circular arc from C to B. The distance from A to C and C to B is equal to $2 \times \pi \times 5 = 10\pi$. Then the difference in the two paths is 0.

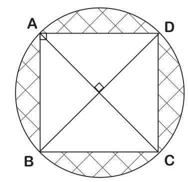
19. Chantelle had two candles, one of which was 32 cm longer than the other. She lit the longer candle at 3 p.m. and lit the shorter one at 7 p.m. At 9 p.m., they were both the same length. The longer one was completely burned out at 10 p.m. and the shorter one was completely burned out at midnight. The two candles burned at different, but constant, rates. What was the sum of the original lengths of the two candles?

Let L, l, R, r represent the length of the longer candle, length of the shorter candle, rate that the longer candle burns at, and rate that the shorter candle burns at, respectively. Then, $L = l + 32$. Since it takes the longer candle 7 hours to burn out, $R = \frac{l+32}{7}$. It takes the shorter candle 5 hours to burn out so $r = \frac{1}{5}$. At 9 pm, the candles are the same length so:

$$\begin{aligned} l + 32 - (6 \times (\frac{l+32}{7})) &= l - (2 \times (\frac{l}{5})) \\ 7l + 224 - 6l - 192 &= \frac{21l}{5} \\ 5l + 160 &= 21l \\ 16l &= 160 \\ l &= 10 \end{aligned}$$

The length of the shorter candle is 10 cm and the longer candle is 42 cm. Thus, the sum of the original lengths is 52 cm.

20. In the diagram, $ABCD$ is a square. If diagonal $AC = 2$ cm, what is the area of the shaded part, in cm^2?



The area of the shaded region is the area of the circle minus the area of the square. $AC = 2$ cm is the diameter of the circle, so the radius is 1 cm.
The area of the circle is $\pi r^2 = \pi(1)^2 = \pi$ cm^2
The square can is divided into 4 equal triangles with base length of 1 cm and height of 1 cm.
Thus, the area of the square is $4 \times \frac{1}{2} \times 1 \times 1 = 2\text{cm}^2$
So the area of the shaded region is $(\pi - 2)\text{cm}^2$.

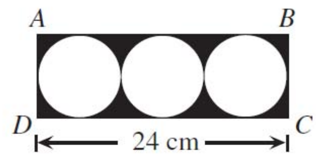
21. At present, the sum of the ages of a father and his son is 33 years. Find the smallest number of years until the father's age is 4 times the son's age.

Let the father's age be represented by x. Then the son's age is $(33 - x)$. In t years, the father will be $x + t$ years old and the son will be $(33 - x) + t$ years old. Then the father's age will be 4 times the son's age when:

$$\begin{aligned} x + t &= 4(33 - x + t) \\ x + t &= 132 - 4x + 4t \\ 5x - 3t &= 132 \end{aligned}$$

Since we want to have the smallest value of t we can test values starting with $t = 1$. If $t = 1$, then $x = 27$. The present ages are 27 and 6 years old. In 1 year, the ages will be 28 and 7 years. Thus, in 1 year the father will be 4 times as old as his son.

22. In the diagram, $ABCD$ is a rectangle, and three circles are positioned as shown. What is the area of the shaded region, rounded to the nearest cm^2?



To find the area of the shaded region, we need to find the area of the rectangle and subtract the area of the three circles from it. Since the three circles are of the same size, they have the same diameter. The sum of their diameters will be equal to the length of the rectangle, so the diameter of each circle is 8 cm. This also tells us that the height of the rectangle is 8 cm. The area of the rectangle is $8 \times 24 = 192\text{cm}^2$. Then the area of the circles is $3 \times \pi r^2 = 3 \times \pi \times (4)^2 = 3 \times 16 \times \pi = 48\pi$.
The area of the shaded region is then $(192 - 48\pi)\text{cm}^2$

23. A sum of money is divided among Allan, Bill and Carol. Allan receives \$1 plus one third of what is left. Bill receives \$6 plus one third of what remains. Carol receives the rest, which amounts to \$40. How much did Bill receive?

After Allan had receives his share of the money, Bill received \$6 plus one-third of the remainder. Then, since Carol receives the rest of the money, she receives two-thirds of the remainder, which is \$40. Therefore, one-third of the remainder is \$20 and thus Bill receives is \$26.

24. In her backyard garden, Gabriella has 12 tomato plants in a row. As she walks along the row, she notices that each plant in the row has one more tomato than the plant before. If she counts 186 tomatoes in total, how many tomatoes are there on the last plant in the row?

Suppose that Gabriellas twelve tomato plants had 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 tomatoes. Then she would have 78 tomatoes in total. We know that she has 186 tomatoes, so there are 108 tomatoes unaccounted for. Since the number of tomatoes on her plants are twelve consecutive whole numbers, then her plants must each have an equal number of additional tomatoes. There will be 9 additional tomatoes on each plan from our initial assumption since $108 \div 12 = 9$. Therefore, the plants have 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, and 21 tomatoes. The last plant then has 21 tomatoes.