UNIVERSITY OF WATERLOO
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
March 7, 2012
Introduction to Graph Theory

What is graph theory?

Graph theory is the branch of mathematics that studies the properties of linear graphs. It uses diagrams (graphs) to study arrangements of objects and the relationship between them.

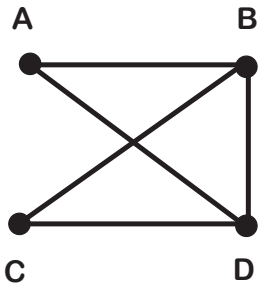
What is a graph?

A graph is defined to be a set of points (called **vertices**) and a set of lines (called **edges**) that connect the vertices.

The vertices of a graph can represent people, places, or things and the edges between vertices represent the relationship between them.

We can represent the edge between vertex A and vertex B by (A,B).

Example: Let G be the graph shown below with vertices {A,B,C,D}. List all the edges of G.



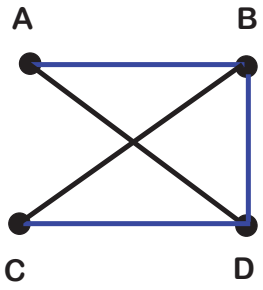
Solution:
{(A,B), (A,D), (B,D), (B,C), (C,D)}

A **path** starts at a vertex, then travels along an edge to another vertex, and continues this pattern a finite number of times until it reaches a designated vertex, never visiting the same vertex twice. We represent the path by listing the sequence of vertices.

A graph is **connected** if there exists a path between any two pairs of vertices in the graph.

Example:

Using the graph G from example 1, find a path from vertex A to vertex C.



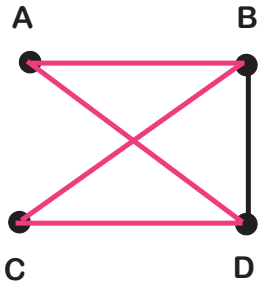
Solution:
{A,B,D,C}

Is the graph connected?

Yes, the graph is connected since there exists a path between any pair of vertices.

A **cycle** is similar to a path but it starts and ends at the same vertex.

Example:
Using the graph G from example 1, find a cycle that starts (and ends) at vertex A.

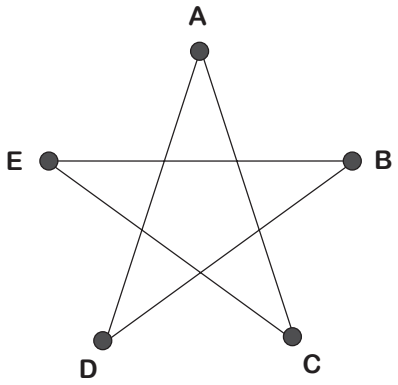


Solution:
{A,D,C,B,A}

Do you notice a relationship between the number of edges in a cycle and the number of vertices in the cycle?

Exercises:

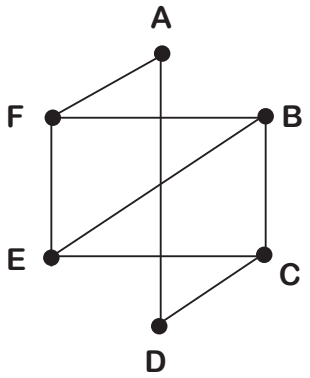
1)



The points of a star are labeled A,B,C,D,E.

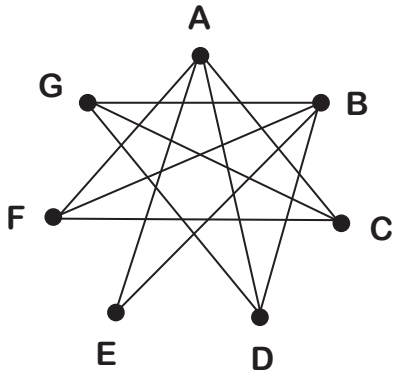
a) List all of the edges of the graph.
b) Determine the shortest path (the path that contains the least number of vertices) from vertex A to vertex B.
c) Is the graph connected?
d) Determine a cycle in the graph that starts at vertex A. Does it contain all of the vertices?

2) a) List all of the edges of the graph.
b) Determine the shortest path from vertex F to vertex D. What is the longest path?
c) Is the graph connected?
d) Determine a cycle in the graph that starts at vertex C. Is it possible to find a cycle, starting at vertex C, that contains all of the vertices?



The graph above has vertices labeled A,B,C,D,E,F.

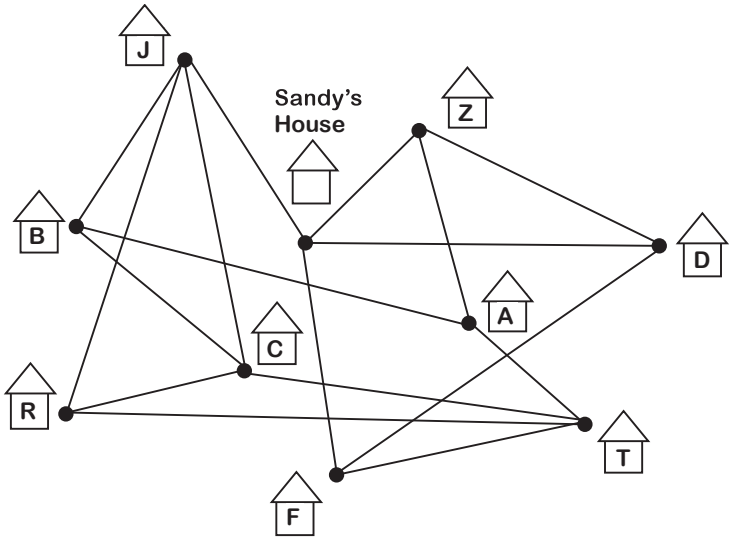
3)



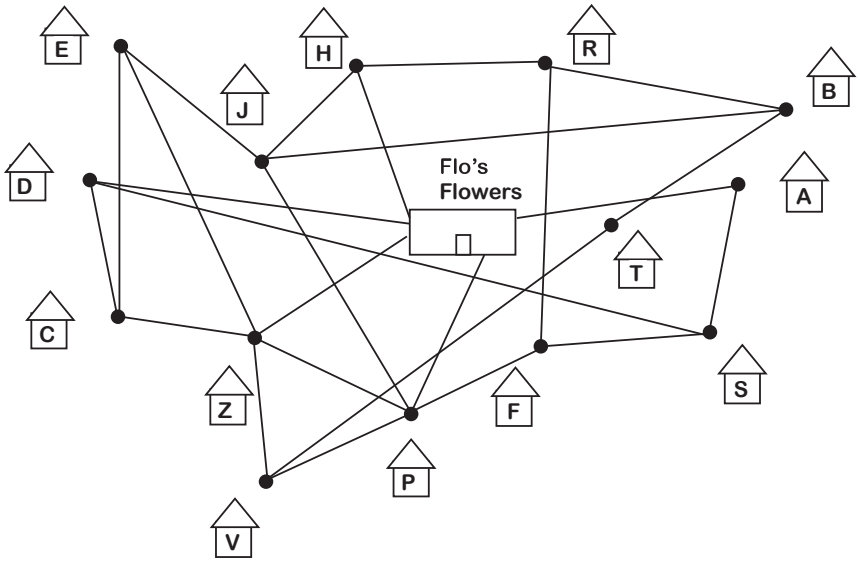
The above graph has vertices labeled A,B,C,D,E,F,G.

a) How many edges does the graph have? List them.
b) Determine the shortest path and the longest path from vertex A to G.
c) Is the graph connected?
d) Determine a cycle in the graph that starts at vertex B. Is it possible to find a cycle, starting at vertex B, that contains all of the vertices?

4) Sandy would like to invite 9 of her friends to her birthday party next weekend. Luckily, all of her friends live in her neighborhood. The graph below shows the location of Sally's house, each of her friend's houses (marked by the first initial of their names), as well as all of the trails that she could take to walk to each person's house. Is it possible for Sally to deliver her invitations without walking along the same path twice? (ie. find a cycle that starts at Sally's house and contains every vertex, or house). If it is possible, what order should she deliver her invitations?

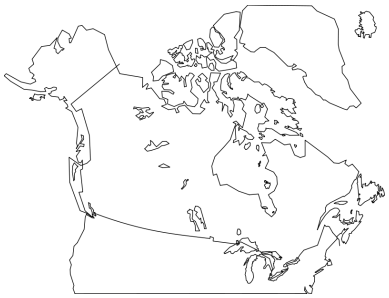


5) Pete recently got a job at Flo's Flowers as a flower delivery boy. This morning he accidentally slept in and is late for his deliveries. Using the graph below, find a route that Pete can take to deliver his flowers and return Flo's Flowers without taking the same road or visiting the same house twice.

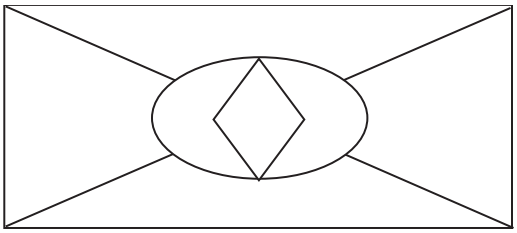


Maps and Colouring:

When you picture a map in your mind, you probably are thinking of a geographical map that shows countries, continents, rivers, oceans, etc. But for mathematicians, maps can consist of lines that divide the plane into different regions. This type of map does not necessarily have to represent anything.



Geographical map



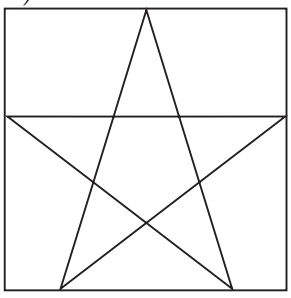
Mathematical map

Note: A geographical map is still a mathematical map, but a mathematical map is not necessarily a geographical map.

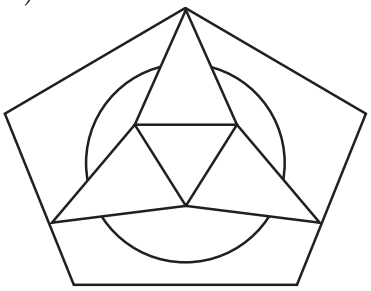
Also note that a mathematical map must cover the whole plane. This means that we consider the region outside the lines to be a region as well. For example, if you're looking at a geographical map of Canada, we would consider the ocean to be a region too.

Examples:
Determine how many regions each of the following maps have.

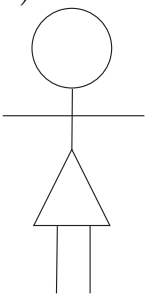
a)



b)



c)



4-Colour Theorem:

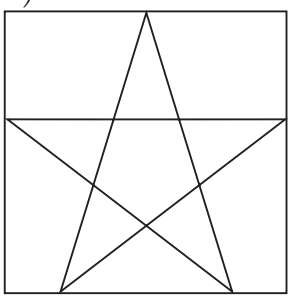
Imagine if all the countries on a map were coloured using the same colour. It would be pretty difficult to differentiate where one country ended and another began! There is a theorem that states that every region on a map can be coloured using at most 4 colours such that no two regions that share an edge are the same colour.

Note: Although we do not allow two regions that share an edge to be coloured using the same colour, two regions that share one vertex may be coloured using the same colour.

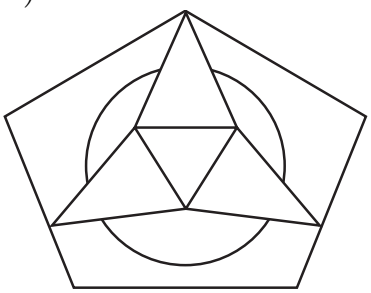
Example:

Colour each of the following using at most 4 colours.
Note: you can use numbers to differentiate the colours if you don't have pencil crayons.

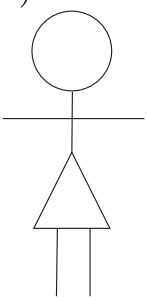
a)



b)

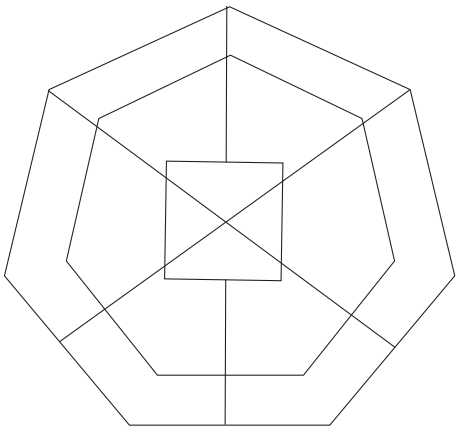


c)



Activity:

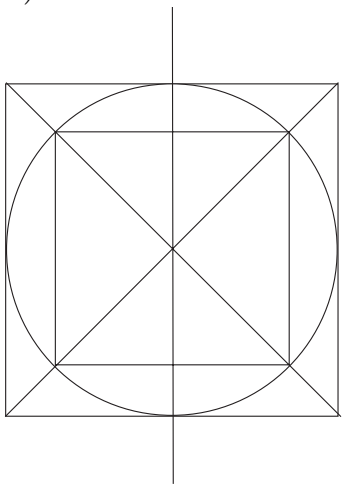
Challenge the person beside you to this colouring game. Using a maximum of four colours, colour the diagram below by alternating taking turns to colour a region. You will be declared the winner if your partner has to use a fifth colour or if you colour the last region.



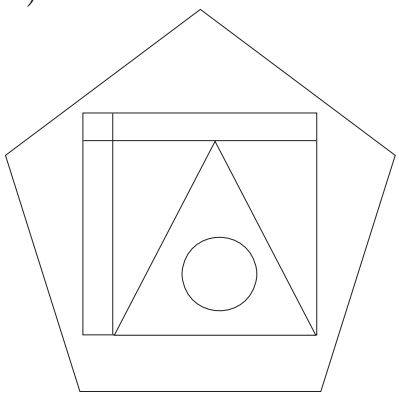
Exercises:

1) Determine how many regions each of the following maps have.

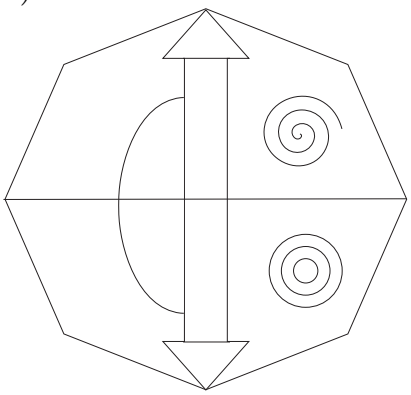
a)



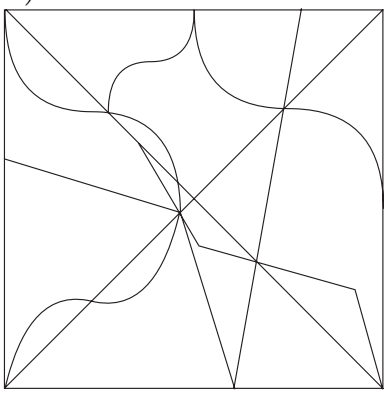
b)



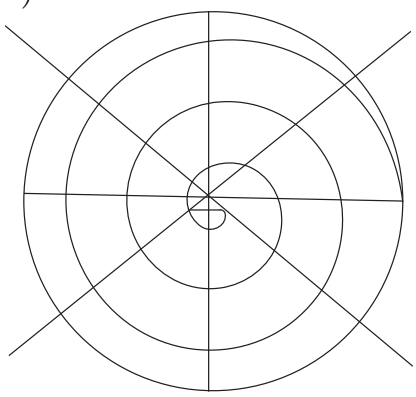
c)



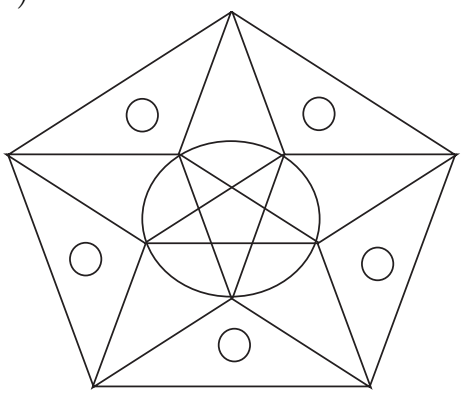
d)



e)



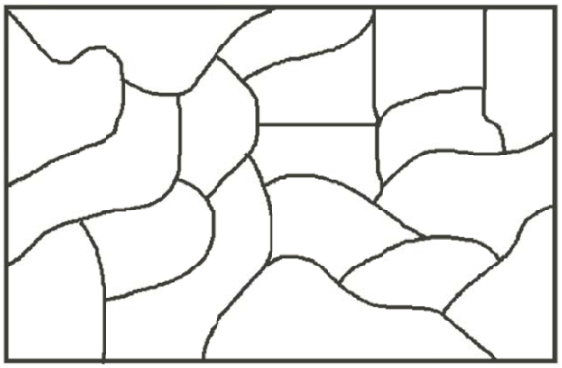
f)



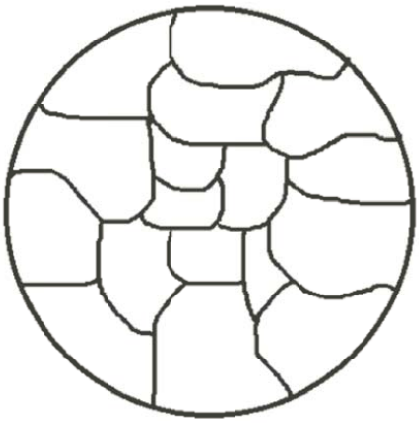
2) Colour each of the maps in exercise 1 using at most 4 colours.

3) Colour each of the following using at most 4 colours.

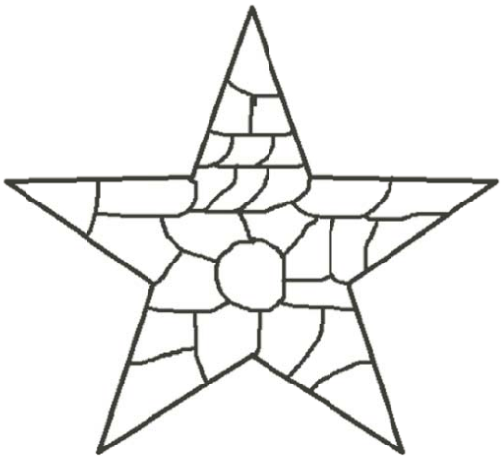
a)



b)



c)



4) Try drawing your own map, then challenge your friends to colour it using 4 colours or less.

5) Can you think of a map that would require only 2 colours to properly colour it?