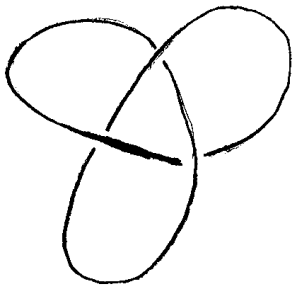
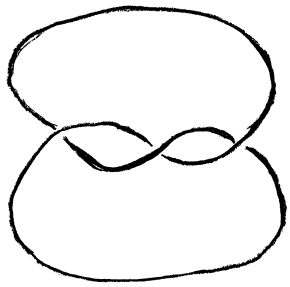
UNIVERSITY OF WATERLOO
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

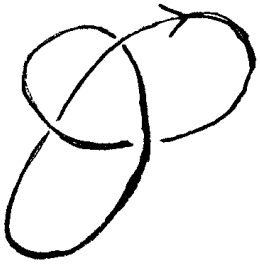
Senior Math Circles
April 4, 2012
Elementary Knot Theory

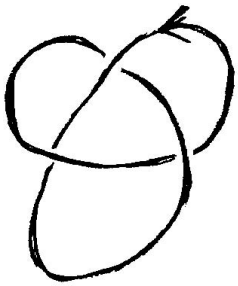
1. Why are the two knots shown below the same?



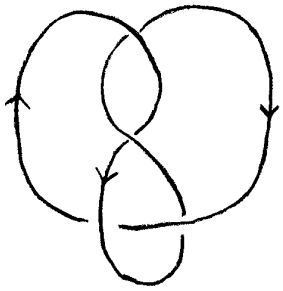


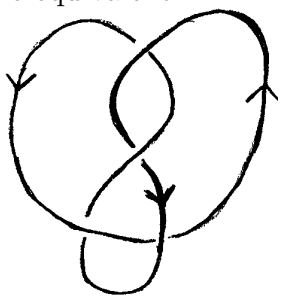
2. Why are the two knots shown below the same?





3. Show that the two knots in the diagrams below are equivalent.





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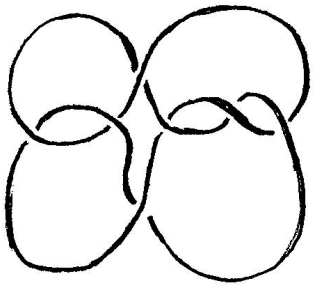
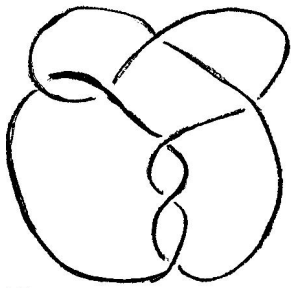
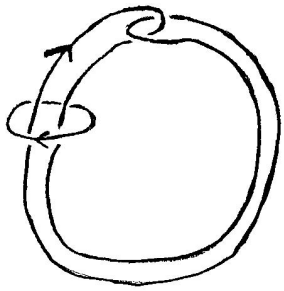
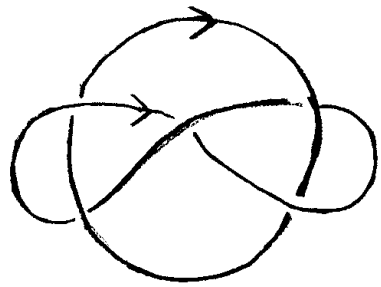


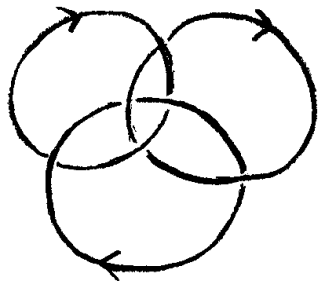


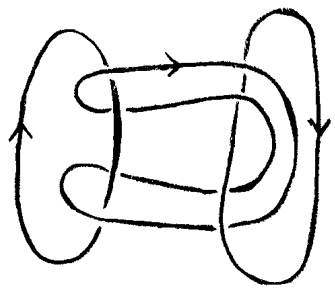
5. Show that the following knots are the same.





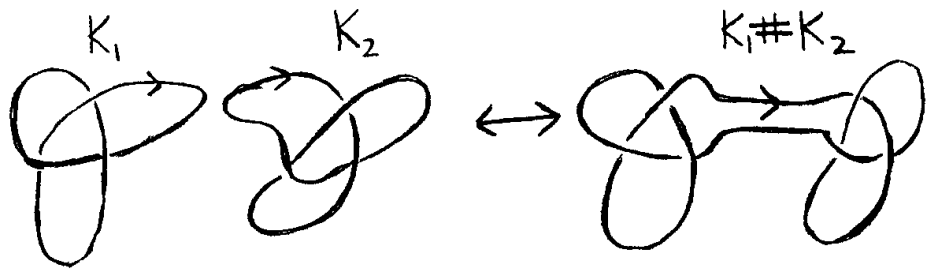
6. Show that the following knots are the same.





Knot Decomposition:

7. Is this well defined?

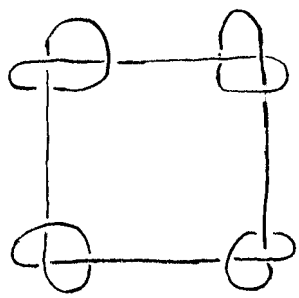


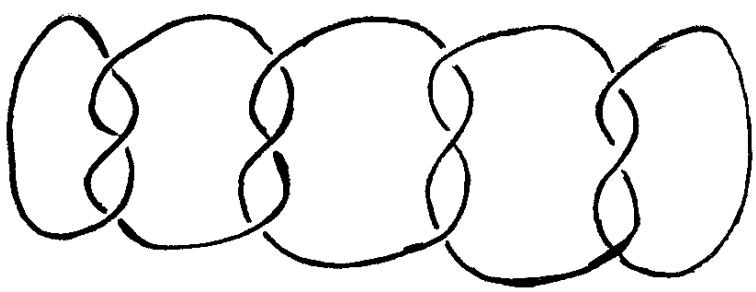
8. Show $K_1 \# K_2 = K_2 \# K_1$.

9. What happens if we add a trivial knot?

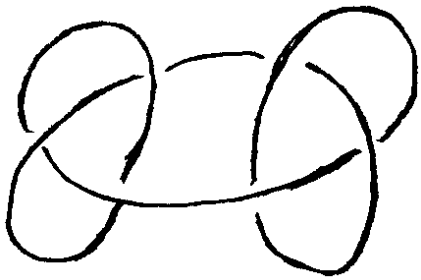
10. Show that a knot can be decomposed into only a finite number of non-trivial knots.

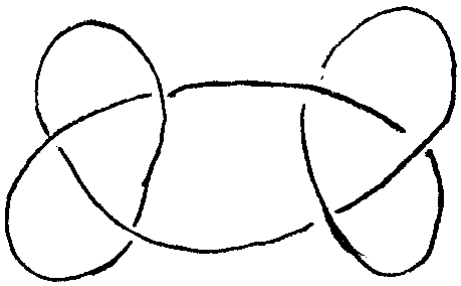
11. Show that the following two knots are equivalent.





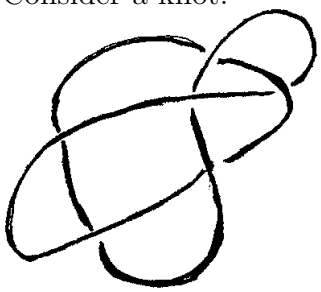
12. Show that the following two knots are **not** equivalent.

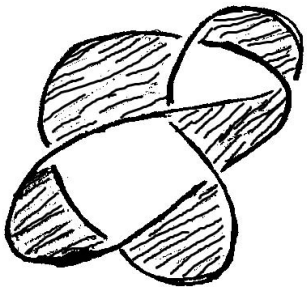




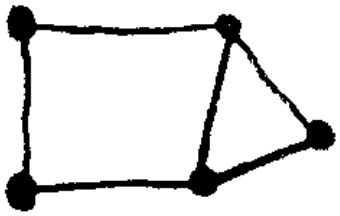
Enumerating Knots

Consider a knot.

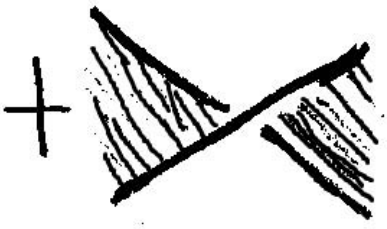


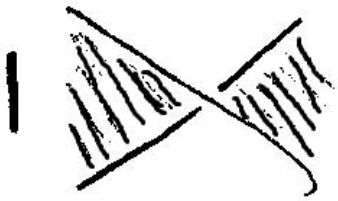


Colour the regions in an alternating fashion, and contract the regions to points, and crossing lines.



Add signs based on:



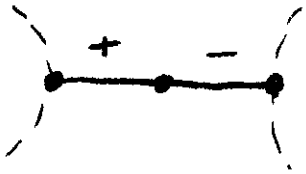


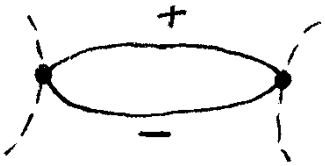
13. Is this well defined?

14. What does the following figure look like on the graph?

15. What does the graph correspond to?

16. How can we simplify the graph containing each of the following?



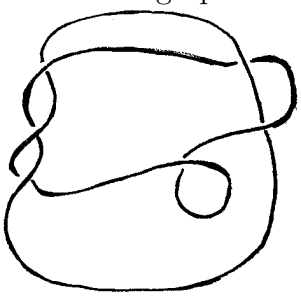


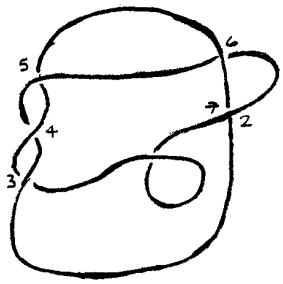
17. Find all non-trivial knots with 0, 1, 2, 3, or 4 edges.

18. Show a regular diagram is alternating if all edges on the graph are the same.

<u>Enumerating Knots 2</u>

Consider a graph:

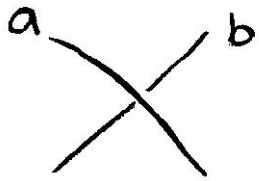




Start at any vertex, labeling it 1, then 2, and so on.

19. Why does every vertex/crossing have two labels?

20. Why do these labels match up odd/even?

For a crossing 

, we write: a b / b -a in the table.

21. Finish the table below for the knot above.

1	2	3	4	5	6	7	8	9	10	11	12
6	−7									−1	2

22. Why is half the information redundant?

23. Show that there cannot exist a know with the following table:

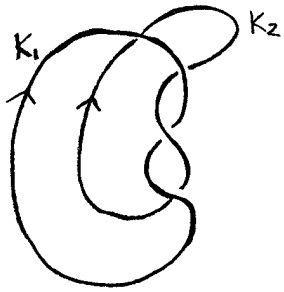
1	2	3	4	5	6	7	8	9	10
8	-5	10	-7	2	-9	4	-1	6	-3

24. Show that if the signs of all the even terms in the bottom row are equal, the we have an alternating knot.

25. How would we extend this idea into links?

<u>Knot Invariants</u>

Consider a link of two knots.



For every crossing of K_1 with K_2, assign:

Add these numbers up, and divide by 2.

This is $lk(K_1, K_2)$.

26. Show that $lk(K_1, K_2)= -1$ in the diagram above.

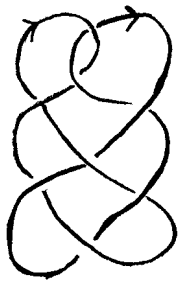
27. What is $lk(K_1, K_2)$ an integer?

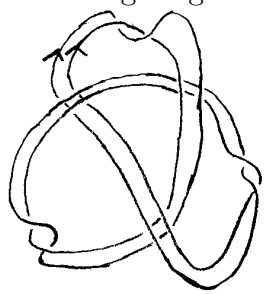
28. Show that $lk(K_1, K_2)$=$lk(K_2, K_1)$

29. Is this an invariant?
(ie. for any link $L - 1 \approx L_2$ made of K_1, K_2, and K_1', K_2', if $lk(K_1, K_2) = lk(K_1', K_2)$)

30. Show $lk(K_1, -K_2)$=-$lk(K_1, K_2)$.
Hint: $-K_2 = K_2$ with orientation reversed.

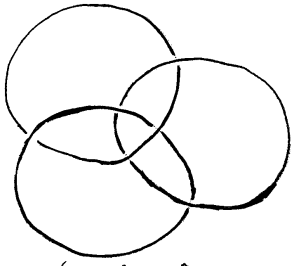
31. Calculate the linking number of each of the following diagrams.





32. Let L^* be the mirror of L. Show $lk(L) = -lk(L^*)$.

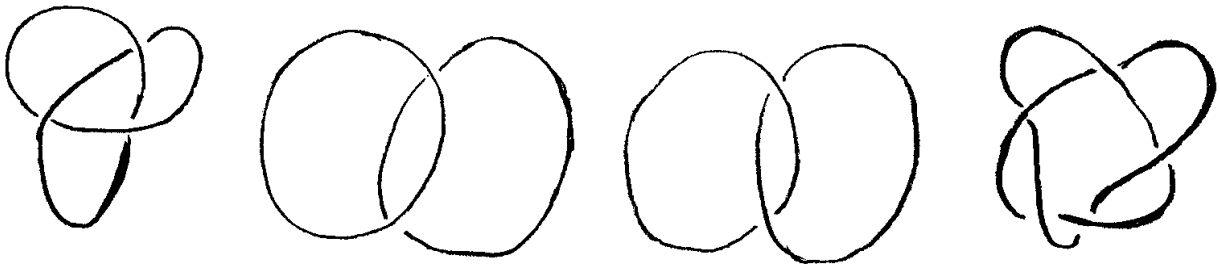
33. Show that $L \neq L^*$ for the diagram below.



(unoriented)

Definition: We say a knot is 3-colourable if we can colour every segment of the diagram such that at crossing points we have 1 or 3 colours, and the knot has at least 2-colours.

34. Which of the following are three colourable?



35. Show that if $K \approx K'$, then K is 3-colourable if and only if K' is.

36. How can we extend this to p-colourable for p-prime.
Hint: Think of colours as numbers a, b, c, ..., modp.

37. Show a knot cannot be 2-colourable but a link can.

38. By using p = 2, 3, and 5, show the above knots are equivalent.