# Intermediate Math Circles
# Wednesday November 14 2012
# Solutions to Problem Set 6

1. Jim calculates the average of a set of $n$ numbers to be 10. Linlin removes the number 2 from the set and recalculates the average of the remaining numbers to be 14. What is the value of $n$?

   **Solution**

   Let $x$ be the original total of all the numbers. The equation that Jim obtains is:

   $$\frac{x}{n} = 10$$
   $$\therefore x = 10n \qquad (1)$$

   Linlin subtracts 2 from the total $x$ and divides by $n-1$, since she has removed the number 2 from the set of numbers:

   $$\begin{aligned} \frac{x-2}{n-1} &= 14 \\ x-2 &= 14(n-1) \\ x-2 &= 14n-14 \\ x &= 14n-12 \qquad (2) \end{aligned}$$

   Since $x = x$ in (1) and (2):

   $$\begin{aligned} 10n &= 14n-12 \\ -4n &= -12 \\ n &= 3 \end{aligned}$$

2. Mr. Galbraith has more than 25 students in his class. He has more than 2 but fewer than 10 boys and more than 14 but fewer than 23 girls in his class. How many different class sizes would satisfy these conditions?

   **Solution**

   Let $b$ be the number of boys in the class, and $g$ be the number of girls in the class. We have the following inequalities from the information:

   $$b+g > 25$$
   $$2 < b < 10$$
   $$14 < g < 23$$

   The maximum number of boys that can be in the class is $b = 9$. The maximum number of girls that can be in the class is $g = 22$. Therefore the maximum on the number of students in the class is $b+g = 9+22 = 31$. We need the class size to be at least 26. So we have the inequality:

   $$26 \le b+g \le 31$$

   If we count this there are 6 possible class sizes. (It is easy to verify that each of the six classes is possible.)

3. The five expressions $2x+1$, $2x-3$, $x+2$, $x+5$ and $x-3$ can be arranged in a different order so that the sum of the first three expressions is $4x+3$ and the sum of the last three expressions is $4x+4$. What is the middle expression in the new list?

**Solution**

Let $S$ be the sum of all the terms, so:

$$S = (2x+1) + (2x-3) + (x+2) + (x+5) + (x-3) = 7x+2$$

Let $A$ be the sum of the first three terms, and let $B$ be the sum of the last three terms, and $m$ be the value of the middle term. Therefore we have the relation:

$$A + B - m = S$$

Now we can substitute all the values in:

$$\begin{aligned} 4x+3+4x+4-m &= 7x+2 \\ m &= x+5 \end{aligned}$$

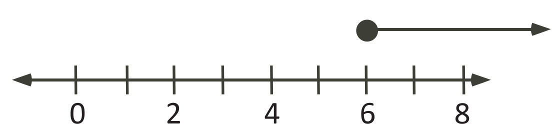
4. Solve $x+2 \leq 3x-10$ and sketch your solution.

**Solution**

$$\begin{aligned} x+2 &\leq 3x-10 \\ 12 &\leq 2x \\ 6 &\leq x \end{aligned}$$

Alternatively,

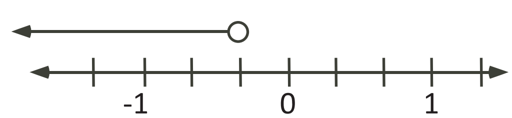
$$\begin{aligned} x+2 &\leq 3x-10 \\ -2x &\leq -12 \\ \text{Dividing both sides by -2:} \quad x &\geq 6 \end{aligned}$$




5. Solve $10+7x < 4x+9$ and sketch your solution.

**Solution**

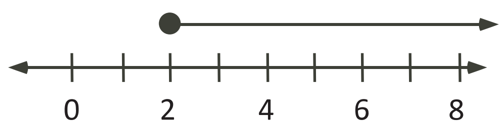
$$\begin{aligned} 10+7x &< 4x+9 \\ 3x &< -1 \\ x &< -\frac{1}{3} \end{aligned}$$



6. Solve $\frac{1}{2}(2+5x) \geq \frac{2}{3}(15-3x)$ and sketch your solution.

**Solution**

$$\begin{aligned}
\frac{1}{2}(2+5x) &\geq \frac{2}{3}(15-3x) \\
\text{Multiplying both sides by 6:} \quad 3(2+5x) &\geq 4(15-3x) \\
6+15x &\geq 60-12x \\
27x &\geq 54 \\
x &\geq 2
\end{aligned}$$




7. How many integer values of $x$ satisfy $\frac{x-1}{3} < \frac{5}{7} < \frac{x+4}{5}$?

**Solution**

Split up the inequality and solve each separately.

$$\begin{aligned}
\frac{5}{7} &< \frac{x+4}{5} \\
\frac{25}{7} &< x+4 \\
\frac{25}{7}-4 &< x \\
-\frac{3}{7} &< x \qquad (1)
\end{aligned}
\qquad\qquad
\begin{aligned}
\frac{x-1}{3} &< \frac{5}{7} \\
x-1 &< \frac{15}{7} \\
x &< \frac{15}{7}+1 \\
x &< \frac{22}{7} \qquad (2)
\end{aligned}$$

$$\therefore -\frac{3}{7} < x < \frac{22}{7} \qquad \text{by (1) and (2)}$$

Since $x$ is an integer, $x \in \{0, 1, 2, 3\}$ and there are 4 integer value solutions.

8. How many positive integers $p$ satisfy $-1 < \sqrt{p} - \sqrt{121} < 1$?

**Solution**

One could solve this inequality in a manner similar to #7 by splitting into two separate inequalities. Alternatively, one could treat this like a three-way inequality - any arithmetic operation done to one term must be done to the other two terms. The solution below illustrates this.

$$\begin{aligned}
& -1 < \sqrt{p} - \sqrt{121} < 1 \\
\text{Simplifying,} \quad & -1 < \sqrt{p} - 11 < 1 \\
\text{Adding 11 to all 3 terms,} \quad & 10 < \sqrt{p} < 12 \\
\text{Squaring,} \quad & 100 < p < 144
\end{aligned}$$

The integers $\{101, 102, 103, \cdots, 142, 143\}$ are in the interval. There are 43 integers in total.

9. If $-4 < x < 6$ then determine $a$ and $b$ in $a < 2x - 5 < b$?

We manipulate the inequality in a manner similar to #8.

$$
\begin{aligned}
& -4 < x < 6 \\
\text{Multiplying by 2,} \quad & -8 < 2x < 12 \\
\text{Subtracting 5,} \quad & -8 - 5 < 2x - 5 < 12 - 5 \\
\text{Simplifying,} \quad & -13 < 2x - 5 < 7
\end{aligned}
$$

But $a < 2x - 5 < b$. Therefore $a = -13$ and $b = 7$.

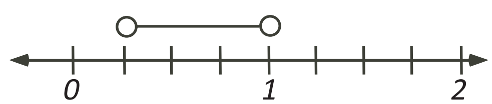
10. What values of $x$ satisfy the inequality $-3 < 5 - \frac{2}{x} < 3$? Sketch your solution.

**Solution**

$$
\begin{aligned}
& -3 < 5 - \frac{2}{x} < 3 \\
& -8 < -\frac{2}{x} < -2 \\
\text{Multiplying by -1,} \quad & 8 > \frac{2}{x} > 2 \\
\text{Dividing by 2,} \quad & 4 > \frac{1}{x} > 1 \qquad (1)
\end{aligned}
$$

Apply the reciprocal property to (1) $\frac{1}{4} < x < 1$.

Therefore the solution is $\frac{1}{4} < x < 1$.



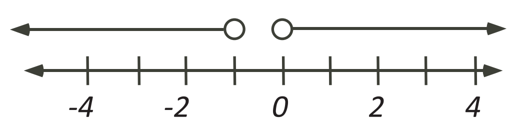

11. Solve $2 - \frac{1}{x} < 3$ and sketch your solution.

**Solution**

$$
\begin{aligned}
2 - \frac{1}{x} &< 3 \\
-\frac{1}{x} &< 1 \\
\frac{1}{x} &> -1
\end{aligned}
$$

The inequality clearly holds for all $x > 0$. If $x < 0$, then $1 < -x$ and $-1 > x$ follows.

Therefore the solution is $x > 0$ or $x < -1$.

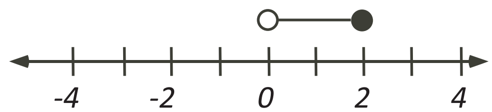


12. Solve $\frac{2}{x}+3 \geq 4$ and sketch your solution.

$$\begin{aligned} \frac{2}{x}+3 &\geq 4 \\ \frac{2}{x} &\geq 1 \end{aligned}$$

If $x>0$, then we have that $2 \geq x$. So the solution for this case is $0<x \leq 2$. Clearly if $x<0$ then there is no solution, since $\dfrac{2}{x}<0$ and cannot be $\geq 1$ at the same time.

Therefore, $0<x \leq 2$.




13. If $\dfrac{\left(\frac{a}{c}+\frac{a}{b}+1\right)}{\left(\frac{b}{a}+\frac{b}{c}+1\right)}=11$ where $a$, $b$, and $c$ are positive integers, how many different ordered triples $(a, b, c)$ are there such that $a+2b+c \leq 40$ is true.

**Solution**

We start with the fact that:

$$\begin{aligned} \frac{\left(\frac{a}{c}+\frac{a}{b}+1\right)}{\left(\frac{b}{a}+\frac{b}{c}+1\right)} &= 11 \\ \frac{\left(\frac{a}{c}+\frac{a}{b}+\frac{a}{a}\right)}{\left(\frac{b}{a}+\frac{b}{c}+\frac{c}{c}\right)} &= 11 \\ \frac{a\left(\frac{1}{c}+\frac{1}{b}+\frac{1}{a}\right)}{b\left(\frac{1}{a}+\frac{1}{c}+\frac{1}{c}\right)} &= 11 \\ \frac{a}{b} &= 11 \\ a &= 11b \end{aligned}$$

Next we can substitute this into our inequality:

$$\begin{aligned} a+2b+c &\leq 40 \\ 11b+2b+c &\leq 40 \\ 13b+c &\leq 40 \end{aligned}$$

Since $a, b, c>0$ and can only be integers, then we observe that the only possible values for $b$ are $1, 2$ and $3$.

For $b=1$, we have

$$\begin{aligned} 13(1)+c &\leq 40 \\ c &\leq 27 \end{aligned}$$

So there are 27 possible positive integer values for $c$ in this case.

For $b=2$, we have

$$\begin{aligned} 13(2)+c &\leq 40 \\ c &\leq 14 \end{aligned}$$

So there are 14 possible positive integer values for $c$ in this case.

For $b=3$, we have

$$\begin{aligned} 13(3)+c &\leq 40 \\ c &\leq 1 \end{aligned}$$

So there is 1 possible positive integer value for $c$ in this case.

If we add up all these possibilities then there are $27+14+1=42$ possible triplets $(a, b, c)$.

14. Jane loves math problems and she receives 1200 new questions from Mr. Shi to complete. From past experience, she knows that:

    - Half of the problems will be difficult to solve, $x$ minutes will be required for each
    - Two-thirds of the remainder are not as hard, she completes these in 40% of the time required for a difficult problem
    - The remaining questions are a snap, she only needs 75% of the time required for a difficult problem

    Jane predicts that the total time required to complete all of the problems must be greater than or equal to 27300 minutes. What is the least amount of time she needs to spend on each difficult problem?

    **Solution**

    The trick is being able to set up the inequality from the information. Let $x$ be the number of minutes to complete a difficult problem, then we have the following relation:

$$\begin{aligned}
\frac{1200}{2} \cdot x + \frac{1200}{2} \cdot \frac{2}{3} \cdot 0.40 \cdot x + \frac{1200}{2} \cdot \frac{1}{3} \cdot 0.75 \cdot x &\geq 27300 \\
600x + 160x + 150x &\geq 27300 \\
910x &\geq 27300 \\
x &\geq 30
\end{aligned}$$

    Therefore, if Jane works as fast as possible, she needs to spend at least 30 minutes on each difficult problem.