Intermediate Math Circles
Wednesday November 14 2012
Equations and Inequalities with One Variable

Problem:

On an island there lives a knight and a dragon. The island is surrounded by water and impossible to escape. The island has 10 magical wells, labelled $\{1, 2, \ldots, 10\}$. Each well contains poisoned water, and if you drink from a well you will die. However, drinking from a well with a higher number will cure you. Well 10 is on a mountain that only the dragon can reach.

The knight and dragon deeply despise each other, like most knights and dragons do. They agree to meet in exactly one month to settle their differences in a game. They will both fill a chalice of whichever liquid they choose, trade chalices, and then drink from the given chalices. They then go their separate ways.

A day after they meet, the dragon dies and the knight lives. Can you explain the strategy the knight used?

Solution:

Using the fact that the island is surrounded by water, the knight fills his chalice with normal water. The dragon expecting that he can cure any poison the knight gives him, drinks from well 10 after their meeting. This causes the dragon to poison himself with the highest well, and therefore without a cure he passes away.

The dragon expecting the knight to have no cure for well 10 water fills a chalice from well 10 and goes to the meeting. The knight beforehand drinks from well 1 and purposely poisons himself. After the knight recieves the chalice from the dragon and drinks from it, he cures himself of the well 1 water.

Inequalities:

The statement $a < b$ means:

- The number a is strictly smaller than the number b.
- On the number line, a must be to the left of b.
- $b = a + r$, where $r > 0$.

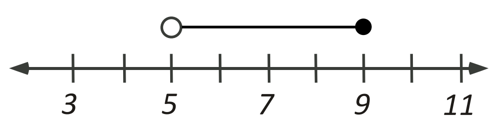
The statement $a \leq b$ means:

- The number a is smaller than or equal to b.
- On the number line, a must be to the left or on b.
- $b = a + r$, where $r \geq 0$.

Compound Inequalities: Given $x > 5$ and $x \leq 9$, we can combine this to form a "compound inequality:

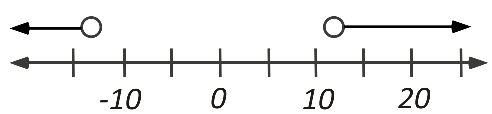
$$5 < x \leq 9$$

This inequality has geometric interpretation when viewed on a number line:



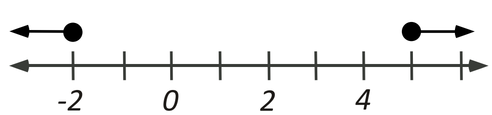
Consider the inequalities $x < -14$ and $x > 12$. We see that there is no number which satisfies these conditions simultaneously. Therefore we conclude that there is no solution to these inequalities.

Drawing the number line, it is clear that there is no intersection for solutions:



Finally, we have compound inequalities which use "or" instead of "and". Given $y \geq 5$ or $y \leq -2$, we only need at least one of the inequalities to be satisfied, not necessarily both at once.

If we draw the number line we see as long as y is not between -2 and 5 this compound inequality is satisfied:



Rules for Inequalities:

1. Adding any number to both sides of the inequality preserves the inequality.

$$\text{If } a < b \text{ then } a + c < b + c$$

Example. Suppose that $a = 4$, $b = 19$ and $c = 16$. What is the relationship between a and b?

$$4 < 19$$
$$4 + 16 < 19 + 16$$
$$20 < 35$$

Example. Suppose that $a = -6$, $b = 12$ and $c = -5$. What is the relationship between a and b?

$$-6 < 12$$
$$-6 - 5 < 12 - 5$$
$$-11 < 7$$

Proof. Remember that if $a < b$ we can rewrite this inequality as the equality:

$$b = a + r, \text{ where } r > 0$$

For equalities we can add any number to both sides and still have the equality be true. If we add c to both sides we get:

$$b + c = (a + c) + r, \text{ where } r > 0$$

Using the fact that $r > 0$ we arrive at:

$$a + c < b + c$$

□

2. Multiplying both sides of an inequality by a positive number preserves the inequality.

$$\text{If } a < b \text{ and } c > 0, \text{ then } ac < bc$$

Example. Suppose that $a = 4$, $b = 16$ and $c = 3$. What is the relationship between a and b?

$$4 < 16$$
$$4 \cdot 3 < 16 \cdot 3$$
$$12 < 48$$

Example. Suppose that $a = -2$, $b = 4$ and $c = 5$. What is the relationship between a and b?

$$-2 < 4$$
$$-2 \cdot 5 < 4 \cdot 5$$
$$-10 < 20$$

Proof. We can rewrite the inequality $a < b$ as:

$$b = a + r, \text{ where } r > 0$$

We can multiply both sides of an equality by the same number and the equality will still be true:

$$bc = (a + r)c$$
$$bc = ac + rc$$

Since $r > 0$ and $c > 0$ then $rc > 0$. Using the fact that rc is positive, we can conclude that:

$$ac < bc$$

□

3. Dividing both sides of an inequality by a positive number preserves the inequality.

$$\text{If } a < b \text{ and } c > 0, \text{ then } \frac{a}{c} < \frac{b}{c}$$

Example. Suppose that $a = 4$, $b = 16$ and $c = 3$. What is the relationship between a and b?

$$4 < 16$$
$$\frac{4}{3} < \frac{16}{3}$$
$$1\frac{1}{3} < 5\frac{1}{3}$$

Example. Suppose that $a = -2$, $b = 4$ and $c = 5$. What is the relationship between a and b?

$$-2 < 4$$
$$\frac{-2}{5} < \frac{4}{5}$$

The same argument can be repeated for the division of c on both sides of the inequality:

Proof.

$$b = a + r, \text{ where } r > 0$$
$$\frac{b}{c} = \frac{a + r}{c}$$
$$\frac{b}{c} = \frac{a}{c} + \frac{r}{c}$$

Since $r > 0$ and $c > 0$ then $\frac{r}{c} > 0$. Using the fact that $\frac{r}{c}$ is positive, we can conclude that:

$$\frac{a}{c} < \frac{b}{c}$$

□

4. Multiplying both sides of an inequality by a negative number changes the direction of the inequality.

$$\text{If } a < b \text{ and } c < 0, \text{ then } ac > bc$$

Example. Consider $a = 4$, $b = 5$ and $c = -2$

$$4 < 5$$
$$4 \cdot -2 > 5 \cdot -2$$
$$-8 > -10$$

Proof. We can rewrite $a < b$ as:

$$b = a + r, \text{ where } r > 0$$

We can then multiply both sides of the equality by the same number and it will still be true:

$$bc = (a + r)c$$
$$bc = ac + rc$$

Since $r > 0$ and $c < 0$ then $rc < 0$. Using the fact that rc is negative, we can conclude that:

$$ac > bc$$

□

5. Dividing both sides of an inequality by a negative number changes the direction of the inequality.

$$\text{If } a < b \text{ and } c < 0, \text{ then } \frac{a}{c} > \frac{b}{c}$$

Example. Consider $a = -2$, $b = -9$ and $c = -3$

$$-2 > -9$$
$$\frac{-2}{-3} < \frac{-9}{-3}$$
$$\frac{2}{3} < 3$$

The same argument can be repeated for the division of c on both sides of the inequality:

Proof.

$$b = a + r, \text{ where } r > 0$$
$$\frac{b}{c} = \frac{a + r}{c}$$
$$\frac{b}{c} = \frac{a}{c} + \frac{r}{c}$$

Since $r > 0$ and $c < 0$ then $\frac{r}{c} < 0$. Using the fact that $\frac{r}{c}$ is negative, we can conclude that:

$$\frac{a}{c} > \frac{b}{c}$$

□

6. If $0 < a < b$, then $a^2 < b^2$

Proof.

$$\begin{aligned} b &= a + r \text{ where } r > 0 \\ b^2 &= (a+r)^2 \\ b^2 &= a^2 + \underbrace{2ar + r^2}_{>0} \end{aligned}$$

Since $a > 0$ and $r > 0$ then $2ar > 0$, as well $r > 0$, therefore $r^2 > 0$. We can then conclude that $2ar + r^2 > 0$ and:

$$a^2 < b^2$$

□

7. If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$

Proof.

$$\begin{aligned} b &= a + r \text{ where } r > 0 \\ \frac{b}{ab} &= \frac{a+r}{ab} \\ \frac{b}{ab} &= \frac{a}{ab} + \frac{r}{ab} \\ \frac{1}{a} &= \frac{1}{b} + \underbrace{\frac{r}{ab}}_{>0} \end{aligned}$$

Since $r > 0$, $a > 0$, and $b > 0$, then $\frac{r}{ab} > 0$. We can conclude that $\frac{1}{a} > \frac{1}{b}$, if $0 < a < b$. □