Intermediate Math Circles
Wednesday November 21 2012
Problem Set 7

1. The point $(a, 2)$ is the point of intersection of the lines with equations $y = 2x - 4$ and $y = x + k$. Determine the value of k.

   Solution

   Since $(a, 2)$ is a point of intersection of the two lines, it satisfies both equations. Substitute $y = 2$ and $x = a$ into the equation $y = x + k$, to obtain $2 = a + k$. Hence $k = 2 - a$.

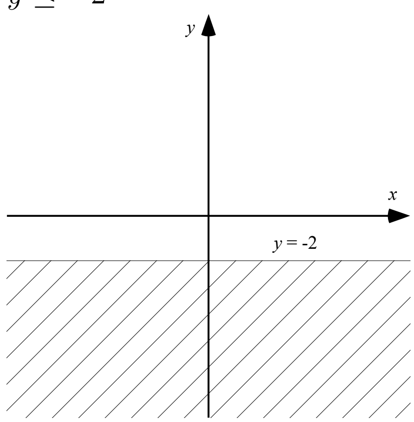
   But $(a, 2)$ also satisfies $y = 2x - 4$, so $2 = 2(a) - 4$. Solving for a gives $a = 3$. Therefore $k = 2 - a = 2 - 3 = -1$.

2. Graph the following regions.

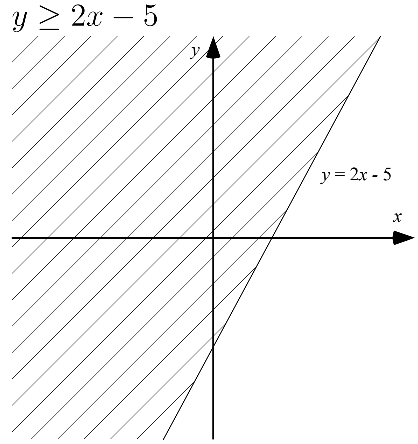
   a) $y \leq -2$ b) $x > 3$ c) $y \geq 2x - 5$ d) $2x + y < 4$

   Solution

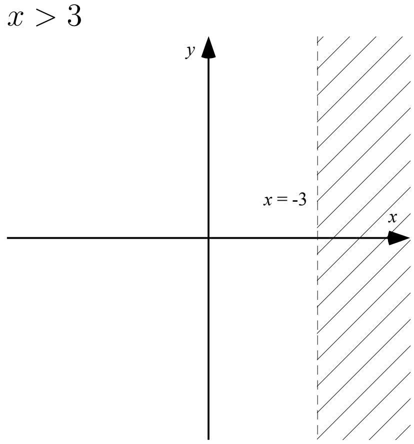
a) $y \leq -2$



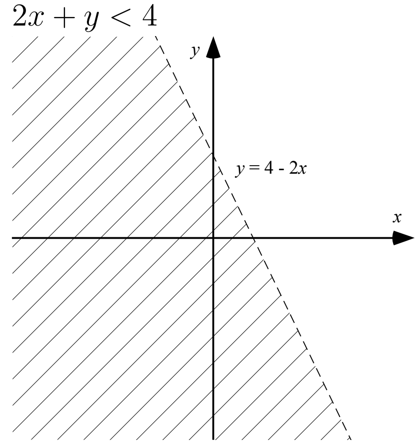
c) $y \geq 2x - 5$



b) $x > 3$



d) $2x + y < 4$



(**N. B.** To decide which side of the line to shade, choose a **test point** clearly not on the line. If it satisfies the inequality, shade in that side of the line. If not, shade the other side. A formal check of a test point is provided in the solution to 3a)

3. To find x-intercepts, set $y = 0$ and solve for x. To find y-intercepts, set $x = 0$ and solve for y. Graph the following regions by finding intercepts.

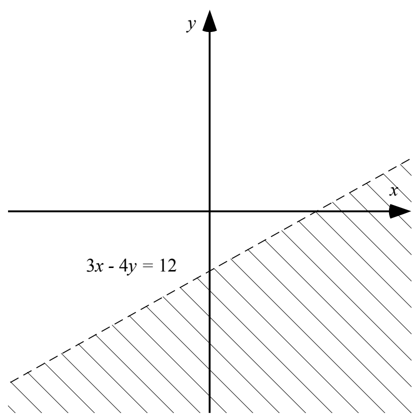
   a) $3x - 4y > 12$

   Solution

   x-intercept: Setting $y = 0$ in $3x - 4y = 12$ gives $3x = 12$ and $x = 4$ follows. Hence the x-intercept is $(4, 0)$.

   y-intercept: Setting $x = 0$ in $3x - 4y = 12$ gives $-4y = 12$ and $y = -3$ follows. Hence the y-intercept is $(0, -3)$.

   Plotting the x- and y-intercepts gives the graph below. To see which side of the line to shade, choose test point $(0, 0)$. The left side of the inequality is $3x - 4y = 3(0) - 4(0) = 0$. But the right side is 12, and clearly $0 \not> 12$. Hence $(0, 0)$ is NOT in the region.



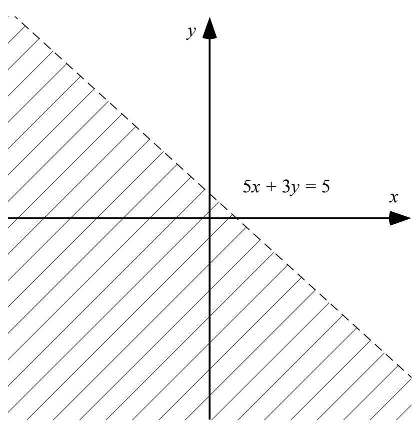
   b) $5x + 3y \leq 5$

   Solution

   x-intercept: Setting $y = 0$ in $5x + 3y = 5$ gives $5x = 5$ and $x = 1$ follows. Hence the x-intercept is $(1, 0)$.

   y-intercept: Setting $x = 0$ in $5x + 3y = 5$ gives $3y = 5$ and $y = \frac{5}{3}$ follows. Hence the y-intercept is $(0, \frac{5}{3})$.

   Plotting the x- and y-intercepts gives the graph.



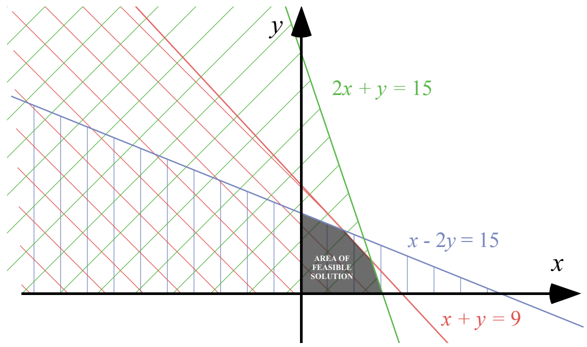
4. Graph the feasible region given the following inequalities:

$$x + y \leq 9$$
$$x + 2y \leq 15$$
$$2x + y \leq 15$$
$$x \geq 0$$
$$y \geq 0$$

Solution

Graph the inequalities using any method. For each line, use test points to figure out which side of the lines to shade. The feasible region is the area in which all points satisfy every inequality (essentially, where all the shadings overlap).

The grey shaded region is the feasible region.



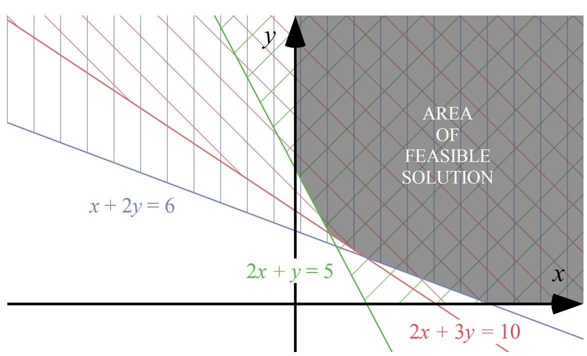
5. Graph the feasible region given the following inequalities:

$$x + 2y \geq 6$$
$$2x + y \geq 5$$
$$2x + 3y \geq 10$$
$$x \geq 0$$
$$y \geq 0$$

Solution

Graph the inequalities using any method. For each line, use test points to figure out which side of the lines to shade. The feasible region is the area in which all points satisfy every inequality (essentially, where all the shadings overlap).

The grey shaded region is the feasible region.



6. The correct formula for converting Celsius temperature C to a Fahrenheit tempeature F is given by $F = \frac{9}{5}C + 32$.

   Andrew does not like arithmetic. So he approximates the Fahrenheit temperature by doubling C and then by adding 30 to get f, the approximate conversion.

   If $f < F$, then the error in the approximation is $F - f$; otherwise, the error in the approximation is $f - F$. Determine the largest possible error in the approximation that Andrew would make when converting Celsius temperatures C with $20 \le C \le 35$.

   Solution

   Let $f = 2C + 30$ represent the approximate conversion. Let $F = \dfrac{9}{5}C + 32$ represent the actual conversion.

   Consider the restriction $20 \le C \le 35$. Observe that within this interval of values for C, $f > F$. Then the error is $f - F = (2C + 30) - (\dfrac{9}{5}C + 32) = \dfrac{C}{5} - 2$.

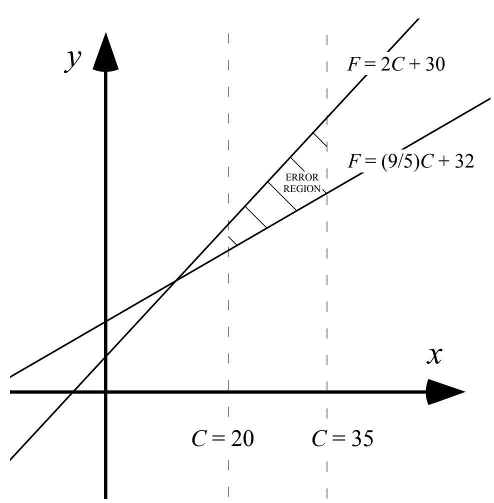
   Manipulating the compound inequality gives

$$\begin{aligned} & 20 \le C \le 35 \\ \text{(Divide by 5)} \quad & 4 \le \frac{C}{5} \le 7 \\ \text{(Subtract 2)} \quad & 2 \le \frac{C}{5} - 2 \le 5 \\ & 2 \le f - F \le 5 \end{aligned}$$

   Hence the largest possible error is 5; Andrew's approximation will be off by at most 5 degrees.

   Alternate Solution

   Sketch the equations $F = \dfrac{9}{5}C + 32$, $f = 2C + 30$, $C = 20$ and $C = 35$ to obtain the graph below.



   In the shaded area (error region), the maximum error is equal to the maximum separation within the region. Clearly, as C goes from 20 to 35, the error increases. So the maximum error occurs at the right endpoint, $C = 35$. The magnitude of this error is 5. Hence the largest possible error is 5 degrees.

7. Gloria is trying to devise a strategy to earn the highest return on her investments. She estimates that investing in real estate yields a 13% annual return on the investment, and the stock market a 17% return. Eeshan does some calculations and advises Gloria to invest at least as much in real estate as in stocks. If she has $\$20,000$ to invest, how should she invest it? Set up the inequalities that satisfy the given conditions.

   Solution

   Let x represent the amount Gloria invests in real estate.

   Let y represent the amount Gloria invests in stocks.

   Since Gloria has up to $20 000 to invest, her two investments can sum up to **at most** this amount. Therefore, $x + y \leq 20000$.

   Eeshan advises her to invest at least as much in real estate as in stocks; hence $x \geq y$.

   Obviously, Gloria cannot invest a negative amount of money, so $x \geq 0, y \geq 0$.

   If Gloria expects to make 13% on real estate returns, she would earn $0.13x$ on real estate. Similarly, since she expects 17% return on stocks, she would earn $0.17y$ from stocks.

   Let R represent her total return. Therefore, $R = 0.13x + 0.17y$, and we want to maximize this amount within the given conditions.

8. Suppose that x and y are positive numbers with

$$\begin{aligned} xy &= \frac{1}{9} \\ x(y+1) &= \frac{7}{9} \\ y(x+1) &= \frac{5}{18} \end{aligned}$$

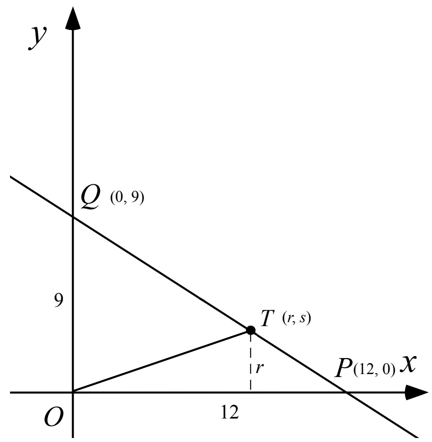
   What is the value of $(x+1)(y+1)$?

   Solution

   Multiply the second and third equations together.

$$\begin{aligned} x(y+1) \cdot y(x+1) &= \frac{7}{9} \cdot \frac{5}{18} \\ (xy)(y+1)(x+1) &= \frac{7}{9} \cdot \frac{5}{18} \\ \frac{1}{9}(y+1)(x+1) &= \frac{7}{9} \cdot \frac{5}{18} \qquad (xy = \frac{1}{9}) \\ (y+1)(x+1) &= \frac{7}{9} \cdot \frac{5}{18} \cdot 9 \\ \therefore (x+1)(y+1) &= \frac{35}{18} \end{aligned}$$

9. The line $y = -\frac{3}{4}x + 9$ crosses the x-axis at P and the y-axis at Q. Point $T(r, s)$ is on line segment $\overline{PQ}$. If the area of $\triangle POQ$ is three times the area of $\triangle TOP$, then what is the value of $r + s$?



P and Q occur at the x and y intercepts respectively. When $x = 0$, the y-intercept occurs when $y = 9$. When $y = 0$, the x-intercept occurs when $x = 12$.

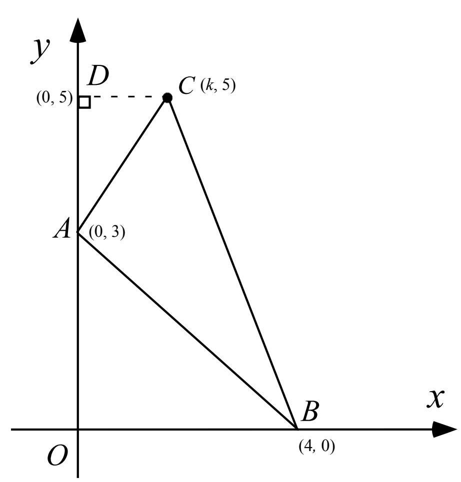
In the diagram, $\triangle QOP$ and $\triangle TOP$ share a common base. If $\triangle TOP$ is one-third the area of $\triangle QOP$, then the height of $\triangle TOP$ must be one third that of $\triangle QOP$ (since the bases are equal and area $= \frac{1}{2}bh$). So $r = \frac{1}{3}(9) = 3$.

Since $T = (r, s) = (3, s)$ lies on the line $y = -\frac{3}{4}x + 9$, $s = -\frac{3}{4}(3) + 9 = \frac{27}{4}$.

Thus $r + s = 3 + \frac{27}{4} = \frac{39}{4}$.

10. A triangle has vertices $A(0, 3)$, $B(4, 0)$, $C(k, 5)$, where $0 < k < 4$. If the area of the triangle $\triangle ABC$ is 8, determine the value of k.

Solution



Construct DC parallel to OB. This forms a trapezoid $DAOB$ with height $OD = 5$, long side $OB = 4$ and short side $DC = k$.

The area of $DAOB$ is

$$\frac{1}{2}(OD)(DC + OB) = \frac{1}{2}(5)(4 + k) \qquad (1)$$

Observe that the trapezoid is composed of $\triangle CDA$, $\triangle CAB$ and $\triangle AOB$. So the area of the trapezoid is also

Area($\triangle AOB$) + Area($\triangle CAB$) + Area($\triangle CDA$) (2)

The areas of the triangles are

- Area of $\triangle AOB = \dfrac{1}{2}(4)(3) = 6$
- Area of $\triangle CAB = 8$ (given)
- Area of $\triangle CDA = \dfrac{1}{2}(AD)(DC) = \dfrac{1}{2}(2)(k) = k$

Set (1) equal to (2) and solve for k

$$\begin{aligned}
\text{Area } DAOB &= 6 + 8 + k \\
\frac{1}{2}(5)(4+k) &= 14 + k \\
5(4+k) &= 28 + 2k \\
20 + 5k &= 28 + 2k \\
3k &= 8 \\
\therefore k &= \frac{8}{3}
\end{aligned}$$