Intermediate Math Circles
Wednesday November 21 2012
Equations and Inequalities with Two Variables

Warm-Up Problem:

51 tigers and 1 sheep are on a magic island that only has grass. Tigers can eat both sheep and grass, sheep can only eat grass. The island is magic because if a tiger eats a sheep that tiger transforms in to a sheep. Tigers enjoy eating sheep over grass, but they enjoy surviving over eating sheep. In other words, the tigers are extremely smart and will only eat sheep if they can live afterwards.

How many tigers and how many sheep are left on the island after everything is resolved?

Consider an island with n tigers and 1 sheep now. What is the equilibrium state for this new island?

Solution:

Consider smaller cases of the situation. If there is 1 tiger and 1 sheep. The tiger automatically eats the sheep, and becomes a sheep since he is guaranteed to survive.

If there are 2 tigers and 1 sheep, no tiger will eat the sheep now, otherwise they would transform and get eaten.

Continuing this logic, the tigers will only want to eat the sheep if there are an odd number of tigers. After that no tiger will eat the sheep, otherwise they would reduce the situation to an odd number of tigers.

Therefore out of the 51 tigers, the first tiger to eat the sheep will transform, and no tiger will want to eat the sheep now for fear of getting eaten. There are 50 tigers remaining and 1 sheep on the island.

We can generalize this further for n tigers where $n \in \mathbb{Z}$, $n \geq 0$.

1. For n even. The island will have n tigers and 1 sheep.

2. For n odd. The island will have $n - 1$ tigers and 1 sheep.

Review:

We noticed some different behaviour when performing certain operations on inequalities versus manipulating a standard equation. We proved these results last time, but we will summarize them below. Assume that we have expressions a and b and some number c we apply to both sides of the inequality.

1. When adding to or subtracting from both sides of an inequality by any $a, b, c \in \mathbb{R}$, the sense of the inequality holds.

$$\text{If } a < b \text{ then } a + c < b + c \text{ and } a - c < b - c.$$

2. When multiplying or dividing both sides of an inequality by c, the sense of the inequality stays the same only for $c > 0$ and flips for $c < 0$.

$$\text{If } a < b \text{ and } c > 0 \text{ then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}.$$

$$\text{If } a < b \text{ and } c < 0 \text{ then } ac > bc \text{ and } \frac{a}{c} > \frac{b}{c}.$$

3. When squaring both sides of an inequality, the sense of the inequality holds provided $0 < a < b$.

$$\text{If } 0 < a < b \text{ then } a^2 < b^2.$$

4. When taking the reciprocal of both sides of an inequality, the sense of the inequality flips provided $0 < a < b$.

$$\text{If } 0 < a < b \text{ then } \frac{1}{a} > \frac{1}{b}.$$

Solving equations and inequalities is the same if we add or subtract the same quantity from both sides. It is also the same if we multiply or divide both sides by positive numbers. In all other cases, we need to take care in performing an operation on both sides of an inequality.

Linear Equations With Two Variables

To talk about inequalities involving two variables, we need to develop some tools for equations in two variables. Linear equations in two variables can be represented geometrically as lines in the Cartesian plane $\mathbb{R}^2$, and can be written in the form:

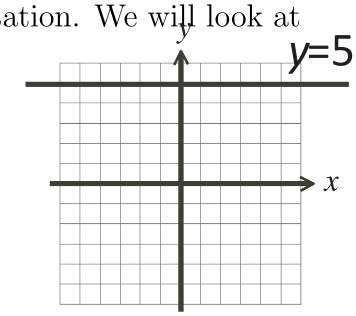
$$y = mx + b$$

where $y, m, x, b \in \mathbb{R}$.

- y is the vertical position.
- m is the slope of the line.
- x is the horizontal position.
- b is the y intercept.

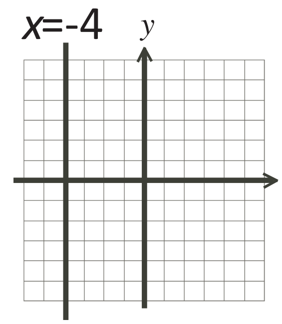
Graphing Lines

Given an equation of a line, we are interested in a geometric interpretation. We will look at some examples to see what happens.



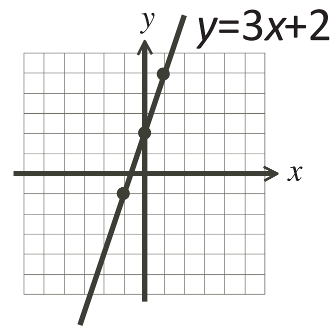
Example 1: Sketch $y = 5$ in $\mathbb{R}^2$.

An equation of the form $y = b$ is a horizontal line b units above the x-axis if $b > 0$ and b units below the x-axis if $b < 0$. In this case, the line is horizontal and 5 units above the x-axis.



Example 2: Sketch $x = -4$ in $\mathbb{R}^2$.

An equation of the form $x = a$ is a vertical line a units right of the y-axis if $a > 0$ and a units left of the y-axis if $a < 0$. In this case, the line is vertical and 4 units to the left of the y-axis.

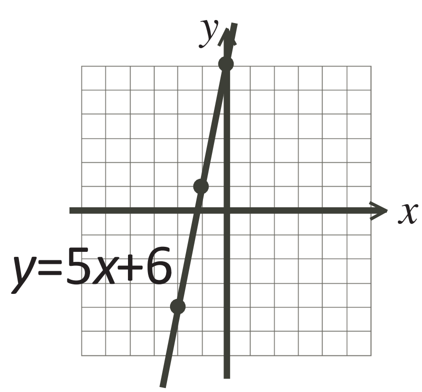


Example 3: Sketch $y = 3x + 2$ in $\mathbb{R}^2$.

The line has y-intercept $b = 2$ so the point (0,2) is on the line. The line has slope $m = 3$. From any point on the line you can go up 3 units and right 1 unit OR go down 3 units and left 1 unit to move to other points on the line. Using this method, (1,5) and (-1, -1) are on the line. Draw the line so that it passes through the points.

Example 4: Two distinct points define a line. Find the equation of the line that passes through $(-1, 1)$ and $(5, 31)$. Sketch the line.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{1 - 31}{-1 - 5} = 5$$



Now we have the equation $y = 5x + b$. To solve for the y-intercept we substitute the co-ordinates of a point on the line. In this case, $x = -1$ and $y = 1$.

$$\begin{aligned} y &= 5x + b \\ 1 &= 5(-1) + b \\ 1 &= -5 + b \\ 6 &= b \end{aligned}$$

Therefore the equation of the line is $y = 5x + 6$. We can verify this by substituting $x = 5$ into $y = 5x + 6 = 5(5) + 6 = 31$, and obtain the correct y-coordinate.

Systems of Equations:

Now we want to look at the intersection of two lines. There are three possibilities:

1. No intersections. This occurs when both lines have the same slope and different intercepts.
2. Infinite intersections. If the two equations represent the same line.
3. One intersection. If the above cases do not apply. The lines intersect in a single point.

Solving for Intersections:

First check to see if the lines are parallel or the lines are multiples of each other. This can usually be done by inspection and will save you time. Otherwise we need to use the following methods:

1. **Substitution:** This is best used when x or y are already isolated.

   Example 5: Find the point of intersection of the lines:

   $$y = -x + 5 \qquad (1)$$
   $$y = 3x - 3 \qquad (2)$$

   Substitute the value of y from equation (1) in to equation (2).

   $$-x + 5 = 3x - 3$$
   $$-4x = -8$$
   $$x = 2$$

   Finally substitute the value of $x = 2$ in to equation (1).

   $$y = -2 + 5$$
   $$y = 3$$

   Therefore out point of intersection is $(2, 3)$.

2. **Elimination:** Works better when lines are in standard form like $Ax + By = C$.

   Example 6: Find the point of intersection of the lines:

   $$6x - 2y = -20 \qquad (1)$$
   $$4x + 2y = -10 \qquad (2)$$

   We can eliminate y if we add equation (1) and equation (2).

   $$6x - 2y + 4x + 2y = -20 - 10$$
   $$10x = -30$$
   $$x = -3$$

   We can now substitute $x = -3$ in to equation (1) to solve for y.

   $$6 \cdot (-3) - 2y = -20$$
   $$-18 - 2y = -20$$
   $$-2y = -2$$
   $$y = 1$$

   Therefore our point of intersection becomes $(-3, 1)$

Solving for Multiple Intersections:

What if we had more than two lines? In order to solve for all intersections, we take all pairwise combinations of lines and solve using either substitution or elimination.

Example 7: Solve for all intersections and graph the following system:

$$\begin{aligned} y &= -x \qquad (1) \\ y &= x \qquad (2) \\ x + 2y &= 12 \qquad (3) \end{aligned}$$

First we will use elimination for equations (1) and (2). We can eliminate x if we add the first two equations:

$$\begin{aligned} (1) + (2) \qquad y + y &= -x + x \\ 2y &= 0 \\ y &= 0 \end{aligned}$$

We substitute $y = 0$ back into equation (1) to solve for the x-coordinate.

$$\begin{aligned} 0 &= -x \\ x &= 0 \end{aligned}$$

Therefore we have a point of intersection of (0,0) for lines (1) and (2).

Next we will solve for the intersection between (1) and (3). We can substitute $-x$ for y into equation (3).

$$\begin{aligned} x + 2 \cdot (-x) &= 12 \\ -x &= 12 \\ x &= -12 \end{aligned}$$

Substitute $x = -12$ back into equation (1) to solve for the y-coordinate.

$$y = -(-12) = 12$$

Therefore the point of intersection for lines (1) and (3) is $(-12, 12)$.

Finally, we will solve for the intersection between (2) and (3). Substitute x for y into equation (3).

$$\begin{aligned} x + 2 \cdot (x) &= 12 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

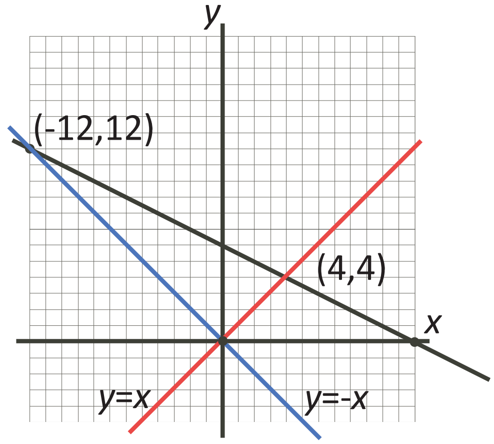
Substitute our value of x back in to equation (2).

$$y = 4$$

Therefore the point of intersection for lines (2) and (3) is $(4, 4)$.

The completed graph is on the next page.

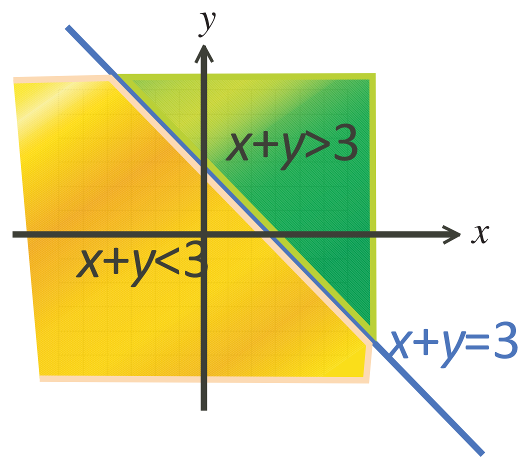
$$y = -x$$
$$y = x$$
$$x + 2y = 12$$



Inequalities in Two Variables:

Consider the line $x + y = 3$. This line divides the xy-plane into three regions:

1. Points that satisfy the equation.
2. Points that satisfy the inequality $x + y < 3$.
3. Points that satisfy the inequality $x + y > 3$.



Example 8: Graph the inequality $x + y \leq 3$.
First, graph $x + y = 3$. Using intercepts, x-intercept=3 and y-intercept=3.
Now pick a point that is clearly not on the line. In this case, $(x, y) = (0, 0)$, the origin is clearly not on the line. If the inequality is true when both sides are evaluated, then the test point is in the region and you would shade that part of the region. Otherwise the point is not in the region and the region is on the other side of the line.
$x + y = 0 + 0 = 0$. Since $0 < 3$, the test point $(0, 0)$ is in the region.
In this example, the solution includes all the points on the line and all points below the line.

System of Linear Inequalities

Here we put it all together. In problems involving a system of inequalities, the solution of the system is the set of points that satisfy *all* the inequalities simultaneously.

Example 9: Graph the solution to the following inequalities:

$$2x + 1.5y \leq 90$$
$$\frac{1}{2}x + \frac{1}{4}y \leq 20$$
$$x \geq 0$$
$$y \geq 0$$

This area containing the solution is called the *feasible region*. We will discuss this in further detail next time.