Intermediate Math Circles
Wednesday November 29 2012
Problem Set 8 Solutions

1. Maximize: $P = 3x + 4y$

   Subject to:
   $$\begin{aligned} 2x + y &\le 6 \\ x + y &\le 4 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

   Sketch the lines:

   $2x + y = 6$

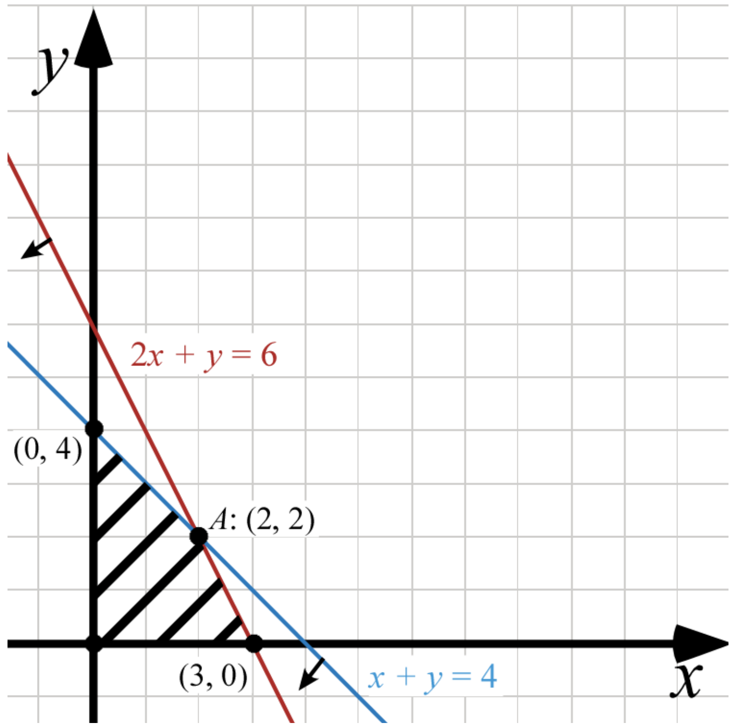
x	y
3	0
0	6

   $x + y = 4$

x	y
4	0
0	4

   Test $(0, 0)$
   $0 < 6$
   In Region

   Test $(0, 0)$
   $0 < 4$
   In Region



Find the intersection points of the two lines (intercepts are easily seen):

$$\begin{aligned} 2x + y &= 6 \quad (1) \\ x + y &= 4 \quad (2) \\ (1) - (2) \qquad x \quad &= 2 \end{aligned}$$

Substituting this back into (2) gives $y = 2$; therefore the intersection is at $A(2, 2)$.

Determine the maximum value of P:

The corners of the feasible region are $(0, 0), (0, 4), (3, 0), (2, 2)$.

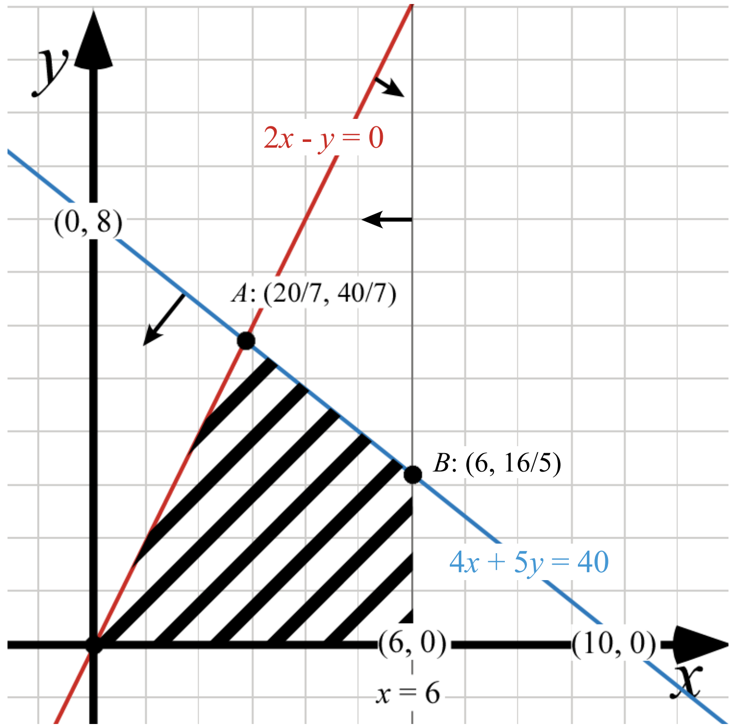
Point	$P = 3x + 4y$
$(0, 0)$	$P = 3(0) + 4(0) = 0$
$(0, 4)$	$P = 3(0) + 4(4) = 16$ (max)
$(3, 0)$	$P = 3(3) + 4(0) = 0$
$(2, 2)$	$P = 3(2) + 4(2) = 14$

Therefore $P = 3x + 4y$ has a maximum value of 16 (subject to the constraints) when $x = 0$ and $y = 4$.

2. Minimize: $C = 2x - 3y$

Subject to:

$$\begin{aligned} 4x + 5y &\leq 40 \\ 2x - y &\geq 0 \\ x &\leq 6 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$



Sketch the lines:

$4x + 5y = 40$

x	y
10	0
0	8

$2x - y = 0$

x	y
2	4
4	8

Test $(0, 0)$
$0 < 40$
In Region

Test $(5, 0)$
$10 > 0$
In Region

Find the intersection points:

$4x + 5y = 40$ (1)
$2x - y = 0$ (2)
Rearrange (2) $y = 2x$
Sub $y = 2x$ into (1) $4x + 10x = 40$
$x = \frac{40}{14}$
$x = \frac{20}{7}$

$4x + 5y = 40$ (3)
$x = 6$ (4)
$4(6) + 5y = 40$
$5y = 16$
$y = \frac{16}{5}$
$y = \frac{16}{5}$

Substituting $x = \frac{20}{7}$ into (2) gives $2(\frac{20}{7}) - y = 0$ and hence $\frac{40}{7} = y$.

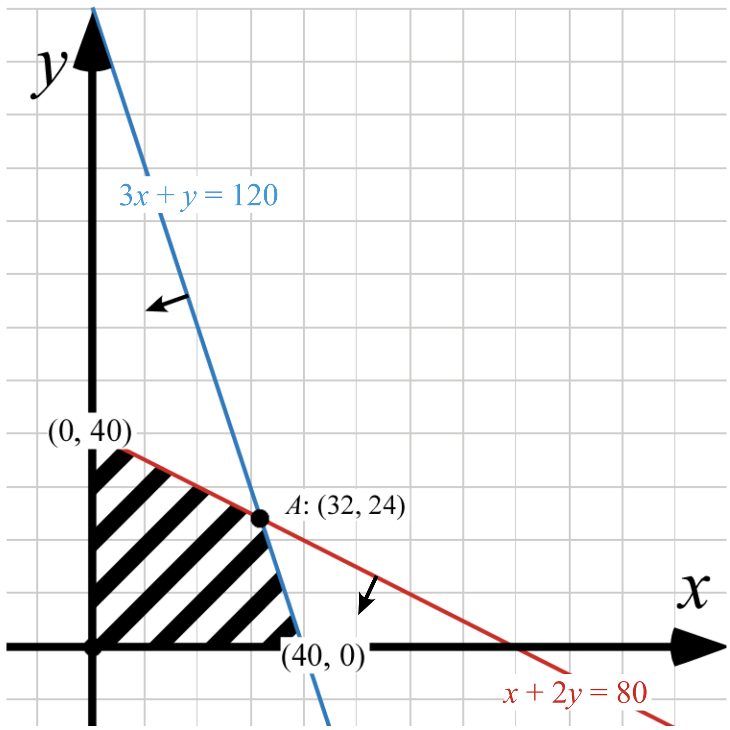
So the points of intersection are $A(\frac{20}{7}, \frac{40}{7})$ and $B(6, \frac{16}{5})$.

Determine the minimum value of C:

Point	$C = 2x - 3y$	Classify
$(0, 0)$	$C = 2(0) - 3(0) = 0$	
$(6, 0)$	$C = 2(6) - 3(0) = 12$	
$(\frac{20}{7}, \frac{40}{7})$	$C = 2(\frac{20}{7}) - 3(\frac{40}{7}) = \frac{-80}{7}$	(min)
$(6, \frac{16}{5})$	$C = 2(6) - 3(\frac{16}{5}) = \frac{12}{5}$	

Therefore $C = 2x - 3y$ has a minimum value (subject to the constraints) of $\frac{-80}{7}$ when $x = \frac{20}{7}$ and $y = \frac{40}{7}$.

3. A tailor has 80 square metres of cotton material and 120 square metres of wool. A suit requires 1 square metre of cotton and 3 square metres of wool. A dress requires 2 square metres of cotton and 1 square metre of wool. How many of each should the tailor make to maximize revenue, if a suit sells for \$110 and a dress sells for \$80?



Define variables:
Let x be the number of suits.
Let y be the number of dresses.
Let R be the revenue.

Objective Function:
$R = 110x + 80y$

Constraints:

$$\begin{aligned} 1x + 2y &\leq 80 && \text{(Cotton needed)} \\ 3x + y &\leq 120 && \text{(Wool needed)} \\ x &\geq 0 && \text{(Make 0 or more)} \\ y &\geq 0 \end{aligned}$$

Graph Equations

$x + 2y = 80$

x	y
80	0
0	40

$3x + y = 120$

x	y
40	0
0	120

Find the intersection of

$$\begin{aligned} & & x + 2y &= 80 && \textcircled{1} \\ & & 3x + y &= 120 && \textcircled{2} \\ & 2 \times \textcircled{2} & 6x + 2y &= 240 && \textcircled{3} \\ & \textcircled{3} - \textcircled{1} & -(x + 2y &= 80) \\ & & 5x &= 160 \\ & & x &= 32 \end{aligned}$$

Substitute $x = 32$ into $\textcircled{1}$. This gives $(32) + 2y = 80$ and hence $y = 24$. Therefore $A(32, 24)$.

Maximize $R = 110x + 80y$

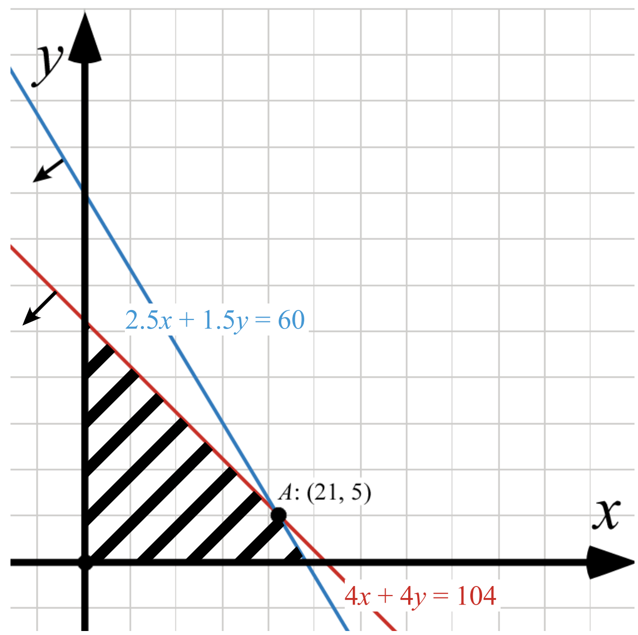
Point	$R = 110x + 80y$	Classify
$(0, 0)$	$R = 110(0) + 80(0) = 0$	
$(0, 40)$	$R = 110(0) + 80(40) = 3200$	
$(40, 0)$	$R = 110(40) + 80(0) = 4400$	
$(32, 24)$	$R = 110(32) + 80(24) = 5440$	(max)

Therefore the tailor will earn a maximum revenue (subject to the constraints) of \$5440 when he/she makes 32 suits and 24 dresses.

4. A company makes two types of calculators, Calculator A and Calculator B. Each calculator must be tested after it is assembled. The amount of time required for assembling Calculator A is 4 hours and the amount of time required for assembling Calculator B is also 4 hours. The amount of time for testing Calculator A is 2.5 hours, and the amount of time for testing Calculator B is 1.5 hours. Each week there are 104 working hours for assembling and 60 working hours for testing. If the company makes a profit of \$4 on each Calculator A and \$2.50 on each Calculator B, how many of each should it produce to maximize its weekly profits?

Define variables:
Let x represent the number of Calc A.
Let y represent the number of Calc B.
Let P be the profit.
Objective Function:
$P = 4x + 2.5y$
Constraints:

$$\begin{aligned} 4x + 4y &\le 104 \quad \text{(Assembly time)} \\ 2.5x + 1.5y &\le 60 \quad \text{(Testing time)} \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$



Graph Equations

$4x + 4y = 104$

x	y
26	0
0	26

$2.5x + 1.5y = 60$

x	y
0	40
24	0

Find the intersection of

$$4x + 4y = 104 \quad ①$$
$$2.5x + 1.5y = 60 \quad ②$$
$$② \times 2 \qquad (5x + 3y = 120) \quad ③$$
$$① \times \frac{3}{4} \qquad 3x + 3y = 78 \quad ④$$
$$③ - ④ \qquad 2x = 42$$
$$x = 21$$

Substitute $x = 21$ into ①. This gives $4(21) + 4y = 104$, and $4y = 20$ follows. Hence $y = 5$ and $A(21, 5)$.

Maximize $P = 4x + 2.5y$

Point	$P = 4x + 2.5y$	Classify
$(0, 0)$	$P = 4(0) + 2.5(0) = 0$	
$(0, 26)$	$P = 4(0) + 2.5(26) = 65$	
$(24, 0)$	$P = 4(24) + 2.5(0) = 96$	
$(21, 5)$	$P = 4(21) + 2.5(5) = 96.5$	(max)

Therefore the company earns a maximum profit of \$96.50 (subject to the constraints) when it makes 21 Calculator A and 5 Calculator B.