Intermediate Math Circles
Wednesday November 21 2012
Inequalities and Linear Optimization

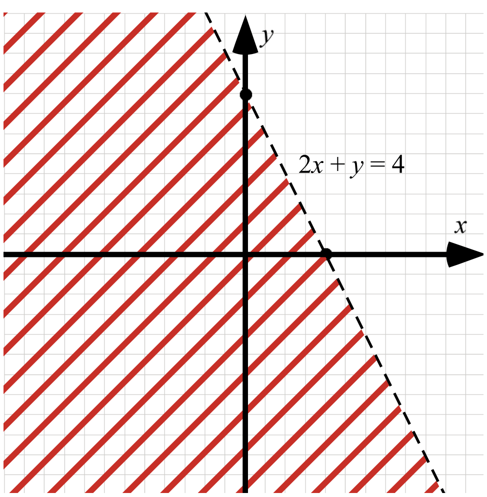
Review: Our goal is to solve systems of linear inequalities and to graph the regions corresponding to them. We will connect this topic to linear optimization.

To graph a linear inequality:

1. Graph the corresponding linear inequality. To do this, find any two points on the line, plot them and draw a line passing through them. The x and y intercepts of the line often work well for the two points. If the inequality is a strict inequality ($<$ or $>$), the line is drawn as a dashed line.
2. Select a test point that is clearly not on the line to substitute into the inequality.
   (a) If the test point satisfies the inequality, shade the region that includes the test point.
   (b) If the test point does not satisfy the inequality, shade the region that does not include the test point.

Example 1: Sketch the inequality $2x + y < 4$.

Since the inequality is strict, a dashed line is drawn.



First, graph $2x + y = 4$ using any method (intercept method used here).

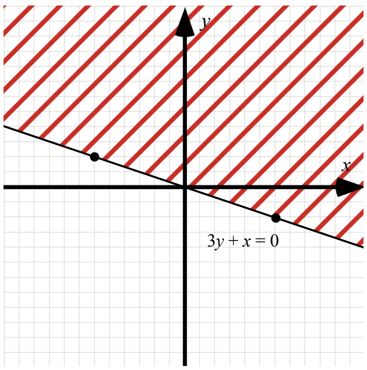
x	y
2	0
0	4

Test the point $(0, 0)$. LS $= 2(0) + (0) = 0 < 4 =$ RS.

Hence $(0, 0)$ is in the region, so shade below the line.

Example 2: Sketch the inequality $3y + x \geq 0$.

Since it is not a strict inequality, we use a solid line.



Graph $3y + x \geq 0$ using any method.

x	y
0	0
-3	1
1	-3

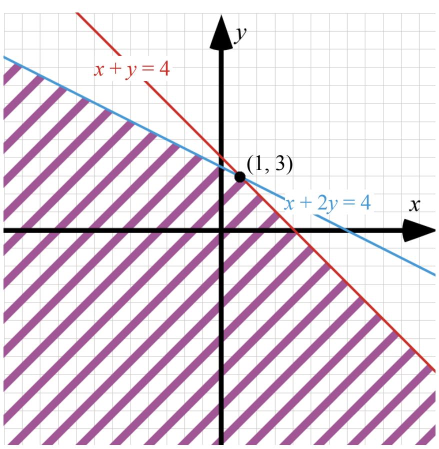
Test the point $(5, 0)$. LS $= 3(0) + (5) = 5 > 0 =$ RS.

Hence $(5, 0)$ is in the region, so shade above the line.

To graph more than one linear inequality, we graph each inequality as before. The *feasible region* is the part of the graph in which each point satisfies every inequality at the same time. We may also be interested in the points that make up the corners of the feasible region. These points could be where lines cross the x or y axes, or intersections of pairs of lines.

Example 3: Sketch the feasible region of

$$x + y \leq 4$$
$$x + 2y \leq 7$$



Graph $x + y = 4$ and $x + 2y = 7$ using any method.

x	y
4	0
0	4

x	y
7	0
0	3.5

Test $(0, 0)$

LS $= x + y$, RS $= 4$
LS $= 0 < 4 =$ RS

Test $(0, 0)$

LS $= x + 2y$, RS $= 7$
LS $= 0 < 7 =$ RS

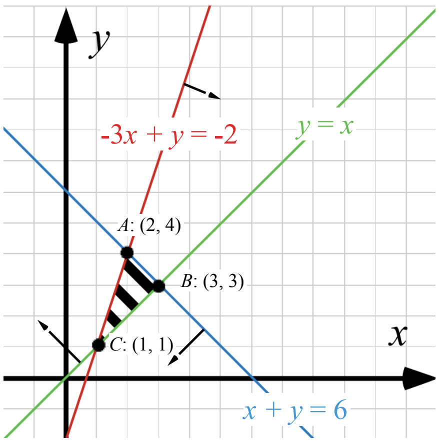
Therefore both test points are in their respective regions. To find the point of intersection, subtract the first equation from the second equation.

$$\begin{aligned} x + y &= 4 \qquad (1) \\ x + 2y &= 7 \qquad (2) \\ y &= 3 \qquad (2) - (1) \end{aligned}$$

Substitute $y = 3$ into (1); $x + (3) = 4$ and hence $x = 1$. Therefore the point of intersection is $(1, 3)$.

Example 4: Sketch the region for the system

$$\begin{aligned} -3x + y &\leq -2 \\ x + y &\leq 6 \\ y &\geq x \end{aligned}$$



Graph $-3x + y = -2$

x	y
0	-2
3	7

Test $(0, 0)$

$0 >$ -2

Not in Region

$x + y = 6$

x	y
0	6
6	0

Test $(0, 0)$

$0 < 6$

In Region

$y = x$

x	y
0	0
5	5

Test $(3, 0)$

$0 < 3$

Not in Region

For the sake of completeness, the intersection points are solved for below.

$$\begin{aligned} -3x + y &= -2 \quad (1) \\ x + y &= 6 \quad (2) \\ &(1) - (2) \\ -3x - x &= -2 - 6 \\ -4x &= -8 \\ \therefore x = 2, y &= 4 \quad (A) \end{aligned}$$

$$\begin{aligned} -3x + y &= -2 \quad (1) \\ y &= x \quad (2) \\ &(2) \to (1) \\ -3x + x &= -2 \\ -2x &= -2 \\ \therefore x = 1, y &= 1 \quad (C) \end{aligned}$$

$$\begin{aligned} y &= x \quad (1) \\ x + y &= 6 \quad (2) \\ &(1) \to (2) \\ y + y &= 6 \\ 2y &= 6 \\ \therefore y = 3, x &= 3 \quad (B) \end{aligned}$$

These points have been indicated on the graph.

Example 5: Maximize $P = 4x - 3y$ subject to the above constraints.

The feasible region gives us all possible pairs of (x, y) that satisfy every constraint (inequality). The intersection points are the corners of the feasible region. There is a theorem that states that the corner points of a feasible region maximize or minimize the *objective function*. In this case, we want to maximize our objective function, which is $P = 4x - 3y$ (you could also have been asked to find the *minimum* value of the function).

To find the corner point which maximizes the objective function, substitute each point and compare the values. The point which gives the highest value is the point which maximizes the objective function.

Point	$P = 4x - 3y$
A $(2, 4)$	$P = 4(2) - 3(4) = -4$
B $(3, 3)$	$P = 4(3) - 3(3) = 3$
C $(1, 1)$	$P = 4(1) - 3(1) = 1$

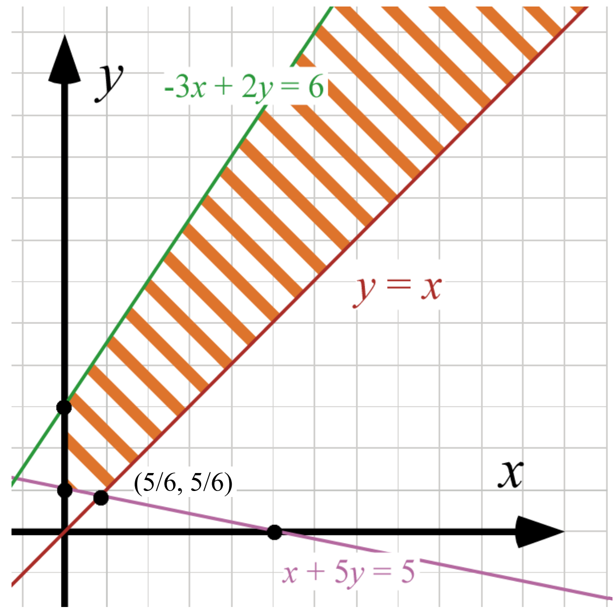
By inspection, B produces the maximum value; hence by the theorem stated above, it maximizes the objective function. Therefore, the maximum value of P (subject to the constraints) is 3 when $x = 3$ and $y = 3$.

Example 6:

Minimize: $C = 4x - 2y$

Subject to:

$$-3x + 2y \leq 6$$
$$x + 5y \geq 5$$
$$y \geq x$$
$$x \geq 0$$
$$y \geq 0$$



Since $x \geq 0$, we want points on the y-axis or to the right of the y-axis. Since $y \geq 0$, we want points on or above the x-axis. Combining the two constraints, we only want points in the first quadrant and on the positive x and y axes.

Graph

$-3x + 2y = 6$

x	y
0	-2
3	7

$x + 5y = 5$

x	y
0	6
6	0

$y = x$

x	y
0	0
5	5

Test:		
$(0, 0)$	$(0, 0)$	$(0, 5)$
$0 < 6$	$0 < 5$	$5 > 0$
In	Not in	In

Intersection Points

The intersection points that occur on the axes are easy to see. To find the intersection point of the lines $y = x$ and $x + 5y = 5$, use substitution.

Since $y = x$, substitute this into $x + 5y = 6$ to obtain $6y = 5$. Solving gives $y = \frac{5}{6}$. Since the intersection point satisfies $y = x$, $x = \frac{5}{6}$ as well. Hence the point is $\left(\frac{5}{6}, \frac{5}{6}\right)$.

It remains to **minimize** the objective function $C = 4x - 2y$.

Point	$C = 4x - 2y$
$(0, 1)$	$C = 4(0) - 2(1) = -2$
$(0, 3)$	$P = 4(0) - 2(3) = -6$
$\left(\frac{5}{6}, \frac{5}{6}\right)$	$P = 4(\frac{5}{6}) - 2(\frac{5}{6}) = \frac{10}{6}$

The minimum value is clearly -6. Hence when $x = 0$ and $y = 3$, the objective function C is minimized (subject to the constraints).

Usually in real world problems, the objective function and constraints are not given. We have to translate the problem into a mathematical model which we can then solve.

In order to do this:

1. Define your variables.
2. State the objective function.
3. Determine appropriate constraints.

Example 7: A farmer is mixing two types of food, brand X and brand Y, for his cattle. Each serving is required to have at least 60 grams of protein and at least 30 grams of fat. Brand X has 15 grams of protein, 10 grams of fat and costs 80 cents per unit. Brand Y contains 20 grams of protein, 5 grams of fat, and costs 50 cents per unit. How much of each brand should be used so that his cost is minimized while still satisfying the requirements for the cattle?

Solution

1. Let x represent the amount of Brand X to buy.

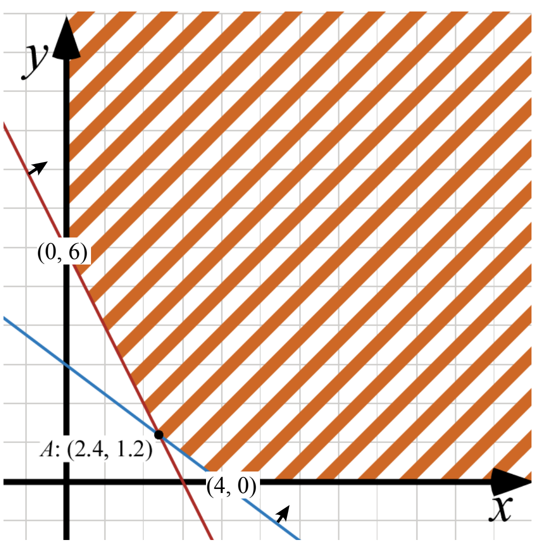
   Let y represent the amount of Brand Y to buy.

   Let C represent the cost of purchasing the food.

2. Objective Function: $C = 0.8x + 0.5y$

3. Constraints:

$$\begin{aligned} 15x + 20y &\geq 60 && \text{(Protein amount)} \\ 10x + 5y &\geq 30 && \text{(Fat amount)} \\ x &\geq 0 && \text{(Cannot buy negative amounts)} \\ y &\geq 0 \end{aligned}$$



Graph

$15x + 20y = 60$

x	y
4	0
0	3

Test $(0,0)$

$0 \not\geq 60$

Not in Region

$10x + 5y = 30$

x	y
3	0
0	6

Test $(0,0)$

$0 \not\geq 30$

Not in Region

It remains to find the points of intersection/corner points which are used to determine the minimize cost. The intercepts are easily seen; it remains to find the intersection of the two lines:

$$10x + 5y = 30 \qquad (1)$$
$$15x + 20y = 60 \qquad (2)$$

$$\begin{aligned} 4 \times (1) \qquad & 40x + 20y = 120 \\ (1) - (2) \qquad & -\ 15x + 20y = 60 \\ & 25x \qquad = 60 \\ & x = \frac{60}{25} \\ & \quad = 2.4 \end{aligned}$$

Substituting this value into (1):

$$\begin{aligned} 24 + 5y &= 30 \\ 5y &= 6 \\ y &= 1.2 \end{aligned}$$

Hence the third intersection point is $(2.4, 1.2)$.

Inputting these points into the objective function and comparing:

Point	$C = 0.8x + 0.5y$
$(4, 0)$	$C = 0.8(4) + 2(0) = 3.2$
$(0, 6)$	$C = 0.8(0) + 0.5(6) = 3$
$(2.4, 1.2)$	$P = 0.8(2.4) + 0.5(1.2) = 2.52$

So $(2.4, 1.2)$ produces the minimal value, and hence minimizes the objective function.

Therefore, the minimum cost (subject to the constraints) is \$2.52 when 2.4 units of brand X and 1.2 units of brand Y are used.