



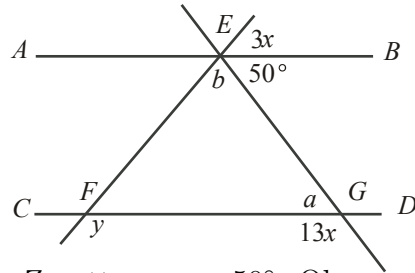
## Intermediate Math Circles

### Wednesday 10 October 2012

### Problem Set 1

1. In the diagram,  $AB$  is parallel to  $CD$ .

Determine the values of  $x$  and  $y$ .



#### Solution

Let  $a = \angle EGF$ ,  $b = \angle FEG$ . Since  $AB \parallel CD$ , by Z pattern,  $a = 50^\circ$ . Observe that  $a$  and the angle  $13x$  form a straight angle. Then,

$$a + 13x = 180$$

$$50 + 13x = 180$$

$$x = 10$$

Similarly, using  $b$ , the  $50^\circ$  angle, and the angle  $3x = 30$ ,

$$b + 50 + 30 = 180$$

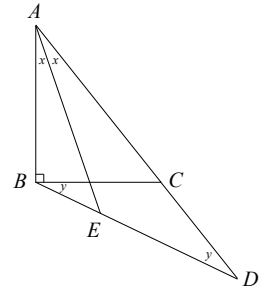
$$b + 80 = 180$$

$$b = 100$$

Since  $y$  is an external angle to  $\triangle EFG$ ,  $y = a + b = 150^\circ$ .

Therefore,  $x = 10^\circ$ ,  $y = 150^\circ$ .

2. Triangle  $ABC$  has a right angle at  $B$ .  $AC$  is extended to  $D$  so that  $CD = CB$ . The bisector of angle  $A$  meets  $BD$  at  $E$ . Prove that  $\angle AEB = 45^\circ$ .



#### Solution

Since  $AE$  bisects  $\angle BAC$ , we can let  $x = \angle BAE = \angle EAC$ .

Since  $CB = CD$ ,  $\triangle BCD$  is isosceles so  $y = \angle CBD = \angle CDB$ .

In  $\triangle ABD$ , by the sum of interior angles of a triangle,

$$2x + (90 + y) + y = 180$$

$$90 + 2x + 2y = 180$$

$$2x + 2y = 90$$

$$x + y = 45$$

In  $\triangle ABE$ , using the sum of interior angles,

$$x + (90 + y) + \angle AEB = 180$$

$$90 + (x + y) + \angle AEB = 180$$

$$90 + 45 + \angle AEB = 180$$

(from above)

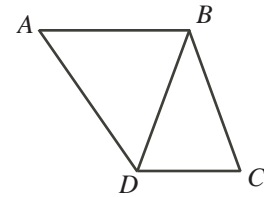
$$45 + \angle AEB = 90$$

$$\angle AEB = 45^\circ$$

(as required)



3. In the diagram,  $AB$  is parallel to  $DC$  and  $AB = BD = BC$ . If  $\angle A = 52^\circ$ , determine the measure of  $\angle DBC$ .



**Solution**

$\triangle ABD$  is isosceles since  $AB = BD$ . Therefore  $\angle BDA = \angle BAD = 52^\circ$ .

Then in  $\triangle BAD$ ,

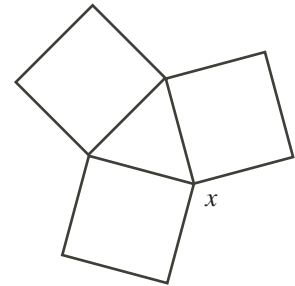
$$\begin{aligned} \angle ABD &= 180^\circ - \angle A - \angle BDA \\ &= 180^\circ - 52^\circ - 52^\circ \\ &= 76^\circ \end{aligned}$$

Since  $AB \parallel DC$ , we have  $\angle BDC = \angle ABD = 76^\circ$ .

Since  $BD = BC$ ,  $\triangle BDC$  is isosceles. Therefore,  $\angle BDC = \angle BCD = 76^\circ$

Therefore, by sum of interior angles of a triangle,  $\angle DBC = 180^\circ - 76^\circ - 76^\circ = 28^\circ$ .

4. The diagram shows three squares of the same size. What is the value of  $x$ ?



**Solution**

In a square, the corner angles are  $90^\circ$ . The triangle is equilateral (all sides equal), so we know all the angles are equal and hence must be  $60^\circ$  each.

If we look at the place where the triangle and two squares meet (where  $x$  is located), we notice it is made up of four angles; two corner angles of a square, one corner angle of a triangle, and  $x$ . These four angles form a complete revolution, so they must sum up to  $360^\circ$ .

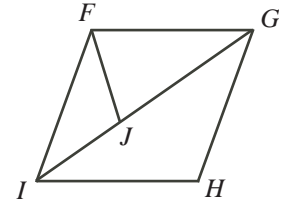
Then,

$$\begin{aligned} x + 90 + 90 + 60 &= 360 \\ x + 240 &= 360 \\ x &= 120^\circ \end{aligned}$$

Therefore the measure of angle  $x$  is  $120^\circ$ .



5. The diagram shows a rhombus  $FGHI$  and an isosceles triangle  $FGJ$  in which  $GF = GJ$ . Angle  $FJI$  equals  $111^\circ$ . What is the measure of angle  $JFI$ ?



**Solution**

Since  $\angle FJI = 111^\circ$  is part of a straight angle with  $\angle FJG$ , we have that  $\angle FJG = 69^\circ$ .

We see that because  $GF = GJ$ ,  $\triangle FGJ$  is isosceles, with equal base angles  $\angle FJG$  and  $\angle GFJ$ , we get  $\angle GFJ = 69^\circ$  and so  $\angle FGJ = 42^\circ$

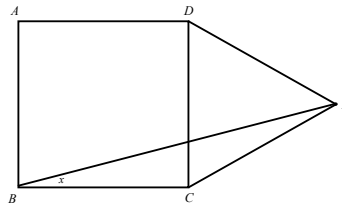
Because  $FG \parallel IH$ ,  $\angle FGI = \angle GIH = 42^\circ$ . Also,  $\triangle IHG$  is isosceles since  $GH = HI$ , so  $\angle IGH = \angle GIH = 42^\circ$

Since  $GH \parallel FI$ ,  $\angle FIG = \angle IGH = 42^\circ$ .

Using  $\triangle FJI$ , we see

$$\begin{aligned} \angle FJI + \angle FIJ + \angle JFI &= 180 \\ 111 + 42 + \angle JFI &= 180 \\ \therefore \angle JFI &= 27^\circ \end{aligned}$$

6.  $ABCD$  is a square. The point  $E$  is outside the square so that  $CDE$  is an equilateral triangle. Find angle  $BED$ .



**Solution**

Since  $ABCD$  is a square,  $BC = CD$ . Since  $\triangle CDE$  is equilateral,  $CD = DE = EC$ . Therefore,  $BC = CD = DE = EC$  and so  $BC = EC$ .

By the properties of a square,  $\angle BCD = 90^\circ$ . By the properties of equilateral triangles,  $\angle DCE = 60^\circ$ . Therefore  $\angle BCE = \angle BCD + \angle DCE = 90 + 60 = 150^\circ$ .

Since  $BC = EC$ ,  $\triangle BCE$  is isosceles. So  $\angle EBC = \angle BEC = x$ . In this triangle, we have

$$\begin{aligned} \angle BCE + x + x &= 180 \\ 150 + 2x &= 180 \\ x &= 15^\circ \end{aligned}$$

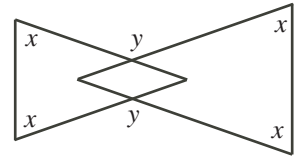
So  $\angle BEC = x = 15^\circ$ .

Note that  $60^\circ = \angle DEC = \angle BED + \angle BEC = \angle BED + 15$ .

Therefore,  $\angle BED = 60 - 15 = 45^\circ$ .



7. The diagram shows two isosceles triangles in which the four angles marked  $x$  are equal. The two angles marked  $y$  are also equal. Find an equation relating  $x$  and  $y$ .



**Solution**

Consider the angles opposite to the angles marked  $y$ . Since they are opposite angles, they are equal to  $y$ .

The quadrilateral formed in the overlap must have angle sum  $360^\circ$ . We know two of the angles are  $y$ .

The other two angles are actually the missing angle of the two isosceles triangles. In the left triangle, this angle is  $180 - 2x$ ; for the triangle on the right, it is also  $180 - 2x$ .

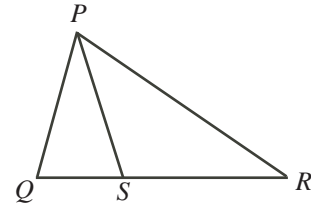
These four angles have to sum to  $360^\circ$ . Therefore,

$$\begin{aligned} y + y + (180 - 2x) + (180 - 2x) &= 360 \\ 2y + 360 - 4x &= 360 \\ 2y &= 4x \\ y &= 2x \end{aligned}$$

$\therefore y = 2x$  is our desired relationship.

8. In the diagram,  $QSR$  is a straight line.

$\angle QPS = 12^\circ$  and  $PQ = PS = RS$ . What is the size of  $\angle QPR$ ?



**Solution**

Let  $\angle SPR = x$ . Then,  $\angle QPR = \angle QPS + \angle SPR = 12^\circ + x$ .

Since  $PS = SR$ ,  $\triangle SPR$  is isosceles and so  $\angle PRS = \angle SPR = x$ . Since  $PS = PQ$ ,  $\triangle PQS$  is isosceles and so  $\angle PQS = \angle PSQ = y$ .

Then

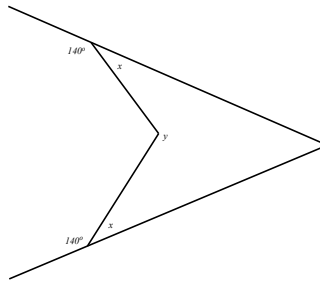
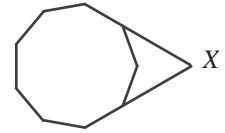
$$\begin{aligned} 12 + y + y &= 180 \\ 2y &= 168 \\ y &= 84^\circ \end{aligned}$$

Since  $QSR$  is a straight line,  $y = \angle PSQ$  is external to  $\triangle PSR$ , so  $84^\circ = y = x + x = 2x$ .

Therefore,  $x = 42^\circ$ .



9. The diagram shows a regular nonagon with two sides extended to meet at point  $X$ . What is the size of the acute angle at  $X$ ?



**Solution**

In a regular nonagon (9 sides), the sum of the interior angles is  $(9 - 2) \times 180^\circ = 1260^\circ$ . Since the figure is regular, all the interior angles are equal.  $\therefore$  each angle is  $\frac{1260}{9} = 140^\circ$ .

Using our diagram, the two extended sides each form a straight angle. One part of each straight angle is the interior angle,  $140^\circ$ . The other part we will call  $x$  must be  $40^\circ$ .

$y$  is part of a revolution; the other part of the revolution is one interior angle of the nonagon,  $140^\circ$ . So  $y = 220^\circ$ .

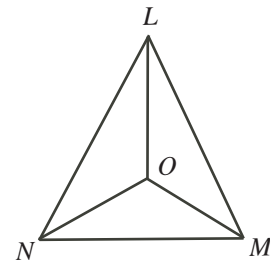
The shape containing the angles  $X, x, y$  is a quadrilateral. The interior sum must therefore be  $360^\circ$ .

So,  $X + x + x + y = 360^\circ$ . Plugging in our values for  $x, y$ , we see

$$X = 360 - 2x - y = 360 - 80 - 220 = 60^\circ$$

Therefore,  $X = 60^\circ$ .

10. The three angle bisectors of triangle  $LMN$  meet at a point  $O$  as shown. Angle  $LMN$  is  $68^\circ$ . What is the size of angle  $LOM$ ?



**Solution**

Since we are using angle bisectors, let  $\angle LNO = \angle ONM = x$ ,  $\angle NLO = \angle OLM = y$ , and  $\angle LMO = \angle OMN = z$ .

But  $68^\circ = \angle LNM = \angle NLO + \angle OLM = 2x$ , so  $x = 34^\circ$ .

We also have  $\angle LON = 180 - (x + y) = 146 - y$ ,  $\angle LOM = 180 - (y + z)$ , and  $\angle NOM = 180 - (x + z) = 146 - z$ .

$\angle LON, \angle NOM,$  and  $\angle LOM$  form a complete revolution.

So,  $\angle LOM = 360 - \angle LON - \angle NOM = 360 - (146 - y) - (146 - z) = 68 + y + z$

Using the entire triangle,

$$\angle LNM + \angle NLM + \angle LMN = 180$$

$$68 + 2y + 2z = 180$$

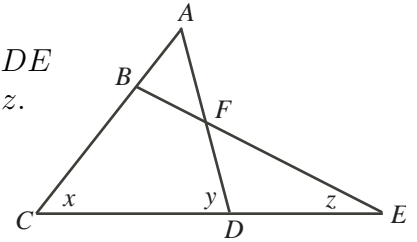
$$2y + 2z = 112$$

$$y + z = 56$$

Therefore, substituting back in, we get  $\angle LOM = 68 + 56 = 124^\circ$ .



11. In the figure shown,  $AB = AF$  and  $ABC$ ,  $AFD$ ,  $BFE$ , and  $CDE$  are all straight lines. Determine an equation relating  $x$ ,  $y$  and  $z$ .



**Solution**

Since  $AB = AF$ ,  $\triangle ABF$  is isosceles, so  $\angle AFB = \angle ABF = a$ .

Since  $\angle AFB$  and  $\angle DFE$  are opposite angles,  $\angle DFE = \angle AFB = a$ .

$\angle ABE$  is external to  $\triangle CBE$ , so  $\angle ABE = \angle ACE + \angle BEC$  and  $a = x + z$  follows. (1)

$\angle ADC$  is external to  $\triangle DFE$ , so  $\angle ADC = \angle DFE + \angle DEC$  and  $y = a + z$  follows. (2)

Substituting (1) into (2) for  $a$ , we obtain  $y = x + z + z$ . Rearranging and simplifying we obtain  $x - y + 2z = 0$ . This is the equation relating  $x$ ,  $y$ ,  $z$ .

12. The angles of a nonagon are nine consecutive numbers. What are these numbers?

**Solution**

In problem 9, we determined that the sum of the interior angles of a nonagon is  $1260^\circ$ .

Order the angles from least to greatest, and let the middle angle (the 5th) be  $x$ . Since they are consecutive numbers, the angles are

$$\{x - 4, x - 3, x - 2, x - 1, x, x + 1, x + 2, x + 3, x + 4\}$$

Summing these angles should give us  $1260^\circ$ . If you add the nine angles, you get  $9x$ . So  $9x = 1260$ .  $\therefore x = 140^\circ$ . This is the fifth angle.

Therefore, the list of angles is  $\{136^\circ, 137^\circ, 138^\circ, 139^\circ, 140^\circ, 141^\circ, 142^\circ, 143^\circ, 144^\circ\}$ .



13. What is the measure of the angle formed by the hands of a clock at 9:10?

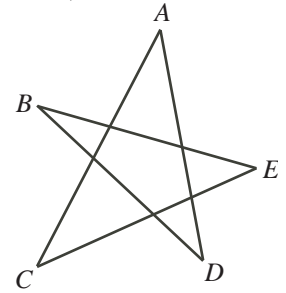
**Solution**

Every minute after the hour, the minute hand moves  $\frac{360}{60} = 6^\circ$  from 12 o'clock. So after 10 minutes, it has moved  $10 \times 6 = 60^\circ$  past 12 o'clock.

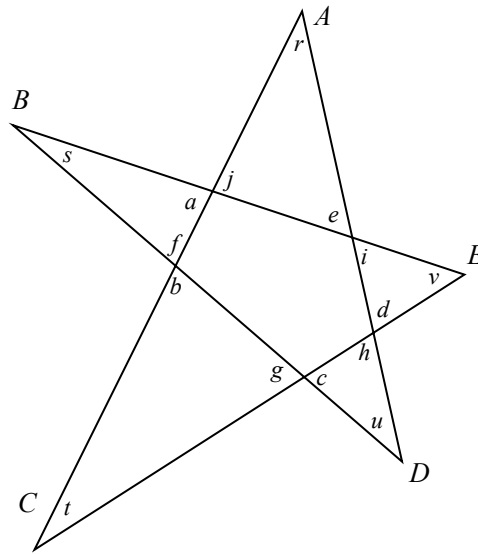
In one hour, the hour hand moves  $\frac{360}{12} = 30^\circ$ . In ten minutes, it will have moved  $\frac{1}{6}$  of this, so it has moved  $\frac{1}{6} \times 30 = 5^\circ$  closer to 12 o'clock. 9 o'clock is located  $90^\circ$  before 12 o'clock, so the hour hand will be  $85^\circ$  before 12 o'clock.

Therefore, the total angle between the hour and minute hand will be  $85 + 60 = 145^\circ$ .

14. Determine the sum of the angles  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  in the five-pointed star shown.



**Solution**



$a, b, c, d, e$  are exterior angles of a pentagon. So they sum to  $360^\circ$ .  $f, g, h, i, j$  are also exterior angles, so they also sum to  $360^\circ$ .

If we add up all the letters in the diagram, we are adding up all the interior angles of five triangles. So the total sum should equal  $5 \times 180 = 900^\circ$ .

Doing this gives,

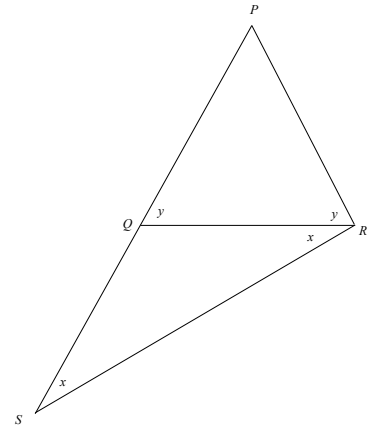
$$\begin{aligned}
 a + b + c + d + e + f + g + h + i + j + r + s + t + u + v &= 900 \\
 (a + b + c + d + e) + (f + g + h + i + j) + r + s + t + u + v &= 900 \\
 (360) + (360) + r + s + t + u + v &= 900 \\
 r + s + t + u + v &= 180 \\
 \therefore A + B + C + D + E &= 180^\circ
 \end{aligned}$$



15. In  $\triangle PQR$ ,  $PQ = PR$ .  $PQ$  is extended to  $S$  so that  $QS = QR$ . Prove that  $\angle PRS = 3(\angle QSR)$ .

**Solution**

Since  $PQ = PR$  and  $QS = QR$ , we can label the diagram as above.

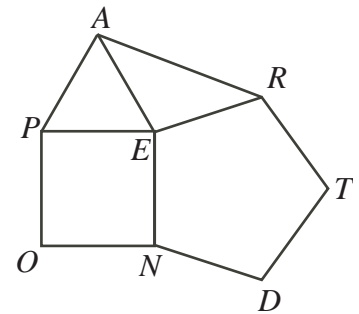


Note that  $\angle SPR = 180 - 2y$ . Using  $\triangle SPR$ , we see the angle sum gives us

$$\begin{aligned} 180 &= \angle SPR + \angle PSR + \angle PRS \\ 180 &= (180 - 2y) + x + (x + y) \\ 180 &= 180 - y + 2x \\ y &= 2x \end{aligned}$$

So  $\angle PRS = x + y = x + 2x = 3x = 3(\angle QSR)$  as required.

16. A regular pentagon is a five-sided figure which has all of its angles equal and all of its side lengths equal. In the diagram,  $TREND$  is a regular pentagon,  $PEA$  is an equilateral triangle, and  $OPEN$  is a square. Determine the size of  $\angle EAR$ .



**Solution**

Since  $\triangle APE$  is equilateral,  $\angle PEA = 60^\circ$ .  
 Since  $OPEN$  is a square,  $\angle PEN = 90^\circ$ .

Since  $TREND$  is a regular pentagon, with interior angle sum is  $540^\circ$ , each angle equals  $540 \div 5 = 108^\circ$ . So  $\angle NER = 108^\circ$ . At E, the angles make a complete rotation, so

$$\begin{aligned} \angle AER &= 360 - \angle PEA - \angle PEN - \angle NER \\ &= 360 - 60 - 90 - 108 \\ &= 102^\circ \end{aligned}$$

Since  $\triangle APE$  is equilateral,  $AE = PE$ . Since  $OPEN$  is a square,  $PE = EN$ . Since  $TREND$  is a regular pentagon,  $EN = ER$ . Therefore  $AE = PE = EN = ER$  and  $\triangle EAR$  is isosceles. It follows that  $\angle EAR = \angle ERA = x$ .

In  $\triangle EAR$ , we then have

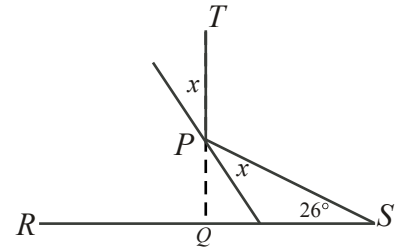
$$\begin{aligned} \angle EAR + \angle ERA + \angle AER &= 180^\circ \\ x + x + 102 &= 180 \\ 2x &= 78 \\ x &= 39 \end{aligned}$$

Therefore,  $\angle EAR = 39^\circ$





17. A beam of light shines from point  $S$ , reflects off a reflector at point  $P$ , and reaches point  $T$  so that  $PT$  is perpendicular to  $RS$ . What is the value of  $x$ ?



**Solution**

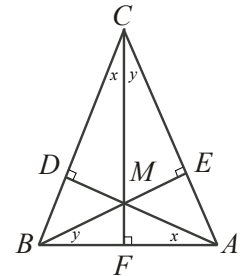
Extend  $TP$  to  $RS$ , intersecting  $RS$  at the point  $Q$  as in the diagram. Then  $\triangle PQS$  is a right triangle.

Since  $\angle TPS$  is exterior to  $\triangle PQS$ ,  $\angle TPS = 90 + 26 = 116^\circ$ .

Since the reflector forms a straight line, the two angles marked  $x$  and  $\angle TPS$  form a straight angle. Then

$$\begin{aligned} \angle TPS + x + x &= 180^\circ \\ 116 + 2x &= 180 \\ 2x &= 64 \\ \therefore x &= 32^\circ \end{aligned}$$

18. In the diagram, let  $M$  be the point of intersection of the three altitudes of triangle  $ABC$ . If  $AB = CM$ , then what is  $\angle BCA$  in degrees?



**Solution**

Let the three altitudes be  $AD$ ,  $BE$  and  $CF$ . In  $\triangle CFB$  and  $\triangle ADB$ , we have  $\angle CFB = \angle ADB = 90^\circ$ .

Also,  $\angle CBF$  and  $\angle DBA$  are the same angle, so  $\triangle CFB \sim \triangle ADB$ .

$\therefore \angle DAB = \angle FCB = x$ .

Applying the same argument to  $\triangle CFA$  and  $\triangle BEA$ , we get  $\angle FCA = \angle EBA = y$ .

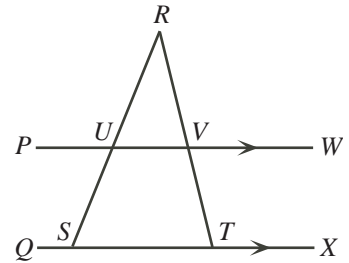
In  $\triangle CDM$  and  $\triangle ADB$ ,

$$\begin{aligned} \angle DCM &= \angle DAB = x \\ \angle CDM &= \angle ADB = 90^\circ \\ \therefore \angle CMD &= \angle DBA \\ CM &= BA \\ \therefore \triangle CDM &\cong \triangle ADB \text{ and } CD = DA \end{aligned}$$

So  $\triangle CDA$  is right isosceles, hence  $\angle DCA = \angle DAC = 45^\circ$ . Therefore  $\angle BCA = 45^\circ$ , since  $\angle DCA = \angle BCA$ .



19. In the diagram,  $PW$  is parallel to  $QX$ ,  $S$  and  $T$  lie on  $QX$ , and  $U$  and  $V$  are the points of intersection of  $PW$  with  $SR$  and  $TR$ , respectively. If  $\angle SUV = 120^\circ$  and  $\angle VTX = 112^\circ$ , what is the measure of  $\angle URV$ ?



**Solution**

Since  $PW \parallel QX$ , we have

$$\begin{aligned} \angle SUV + \angle TSU &= 180^\circ \\ 120 + \angle TSU &= 180 \\ \angle TSU &= 60^\circ \end{aligned}$$

$\angle RTX$  is exterior to  $\triangle RST$ .  $\therefore \angle RTX = \angle SRT + \angle RST$ . (1)

But  $\angle RTX = \angle VTX = 112^\circ$  (same angle, given info)

and  $\angle RST = \angle TSU = 60^\circ$  (same angle)

$\therefore$  substituting in (1), we have

$$\begin{aligned} \angle SRT + 60 &= 112 \\ \angle SRT &= 52^\circ \end{aligned}$$

But  $\angle SRT$  and  $\angle URV$  are the same angle.  $\therefore \angle URV = 52^\circ$ .

20. Three regular polygons meet at a point and do not overlap. One has 3 sides and one has 42 sides. How many sides does the third polygon have? Can you find other sets of three polygons that have this property?

**Solution**

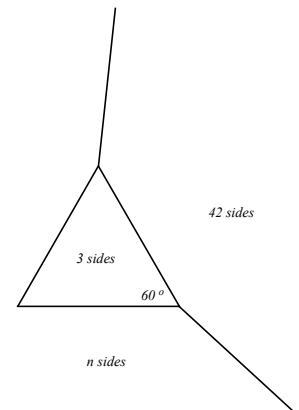
Each angle in a regular 3 sided polygon is  $\frac{180^\circ}{3} = 60^\circ$ .

Each angle in a regular 42 sided polygon is  $\frac{180(42 - 2)}{42} = \frac{1200^\circ}{7}$ .

Each angle in a regular n-gon is  $\frac{180(n - 2)}{n}$ .

The 3 angles form a complete revolution.

$$\begin{aligned} \therefore 60^\circ + \frac{1200^\circ}{7} + \frac{180(n - 2)}{n} &= 360^\circ \\ \frac{180(n - 2)}{n} &= 360^\circ - 60^\circ - \frac{1200^\circ}{7} \\ \frac{180(n - 2)}{n} &= \frac{900^\circ}{7} \\ \frac{(n - 2)}{n} &= \frac{5^\circ}{7} \\ 7n - 14 &= 5n \\ 2n &= 14 \\ n &= 7 \end{aligned}$$



$\therefore$  it is a 7-sided figure.