Intermediate Math Circles
Wednesday 17 October 2012
Geometry II: Side Lengths

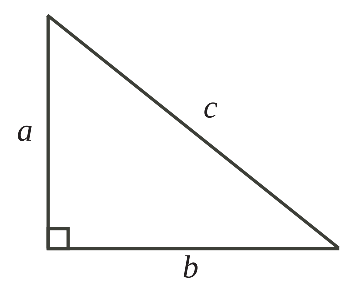
Last week we discussed various angle properties. As we progressed through the evening, we proved many results. This week, we will look at various side length properties and we will prove some results. Some of this material will be familiar and some of this will stretch what you already know.

Problems From Last Week

We will take up four or five problems from last week. Complete solutions can be found on our website at http://www.cemc.uwaterloo.ca/events/mathcircle_presentations.html.

Getting Started

The Pythagorean Theorem:



In a right-angled triangle, the *hypotenuse* is the longest side and is located opposite the 90° angle. In any right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides.

In the triangle illustrated to the right, $a^2 + b^2 = c^2$.

Proofs of The Pythagorean Theorem:

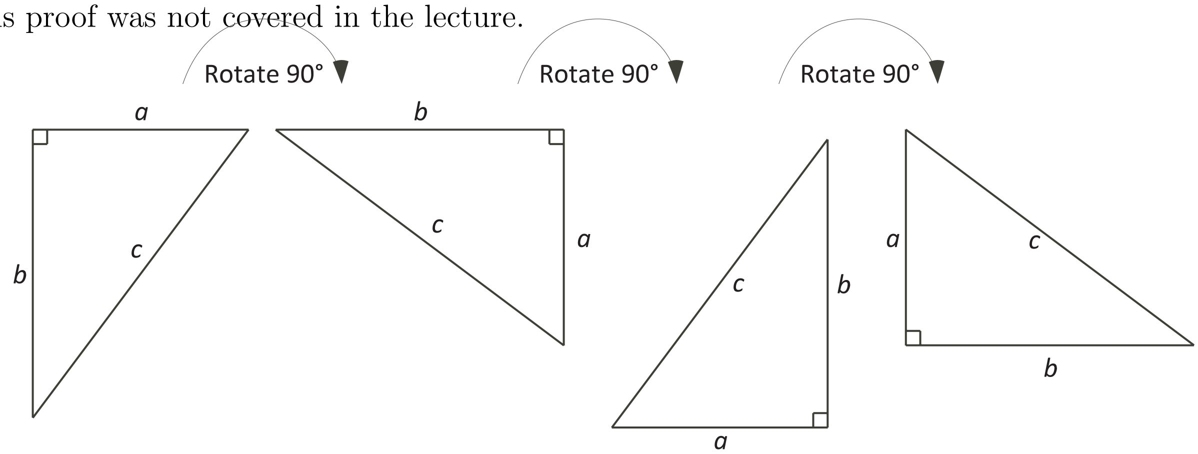
If you do an internet search you will discover many different proofs of the Pythagorean Theorem. If you go to the link http://www.cut-the-knot.org/pythagoras/index.shtml#84, you will find 98 of the proofs grouped together. We will present three proofs here.

Proof #1:

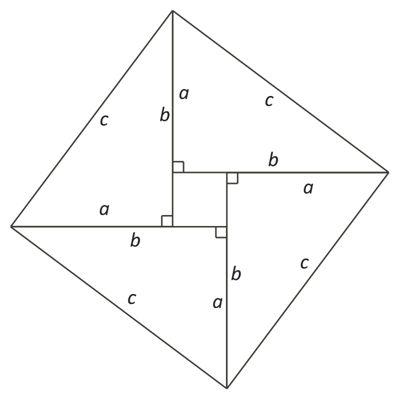
The first proof presented was a visual proof. It will not be included in these notes.

Proof #2:

This proof was not covered in the lecture.



Starting with the leftmost right triangle, rotate 90° to the right to create the second triangle. Rotate the second triangle 90° to the right to create the third triangle and rotate the third triangle 90° to the right to create the fourth triangle. This process creates four congruent right triangles. We will now reposition the four right triangles to create the following figure.



The figure is a square with sides of length c. We can see that the sides are each length c but are the corners 90°? Let the angle between side b and side c be α. Then the angle between side a and side c is $90 - \alpha$. Each corner then consists of an α and $90 - \alpha$. The angle at each corner is $\alpha + 90 - \alpha = 90°$.

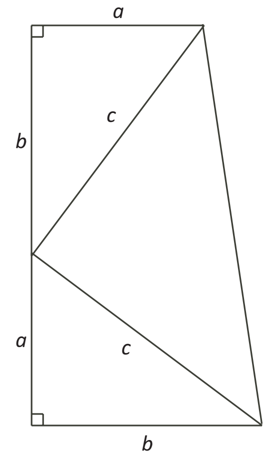
The figure in the centre is a square. Each side is $b - a$ units.

The large square is made up of four congruent triangles and a smaller square. We will construct an equation using area.

$$\begin{aligned}
\text{Area of Large Square} &= \text{Area of 4 triangles} + \text{Area of inner square} \\
c^2 &= 4 \times \left(\frac{a \times b}{2}\right) + (b - a) \times (b - a) \\
c^2 &= 2ab + (b^2 - 2ab + a^2) \\
c^2 &= b^2 + a^2
\end{aligned}$$

This is not the only way to arrange the triangles. A figure can be created with a large outer square of side length $a + b$ and a smaller square of side length c inside along with the four triangles. This will be left as an exercise for the student to pursue.

Proof #3:



This proof is attributed to James Garfield, the twentieth President of the United States. He basically takes the first and fourth triangles from our group of four triangles and stacks them on top of each other as shown.

At the point where the three triangles meet a straight line is formed. Let the angle between side b and side c be α. Then the angle between side a and side c is $90 - \alpha$. The remaining angle between the two sides of length c is $180 - \alpha - (90 - \alpha) = 90°$.

The large figure is a trapezoid that contains three right angled triangles. (The justification that the large figure is a trapezoid is straight forward and is not included here.) As in proof #2 we can form an area equation.

$$\begin{aligned}
\text{Area of Trapezoid} &= \text{Area of Two congruent triangles} + \text{Area of Isosceles triangle} \\
\frac{h \times (a+b)}{2} &= 2 \times \left(\frac{a \times b}{2}\right) + \frac{c \times c}{2} \\
\frac{(a+b) \times (a+b)}{2} &= 2 \times \left(\frac{a \times b}{2}\right) + \frac{c \times c}{2} \\
(a+b)(a+b) &= 2ab + c^2, \qquad \text{after multiplying through by 2} \\
a^2 + 2ab + b^2 &= 2ab + c^2 \\
a^2 + b^2 &= c^2
\end{aligned}$$

A *Pythagorean Triple* is a triple (a, b, c) of positive integers with $a^2 + b^2 = c^2$. What Pythagorean Triples do you know?

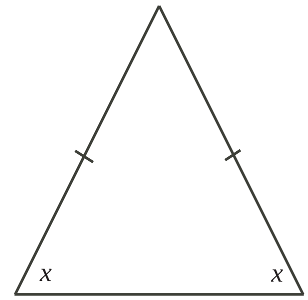
The following chart illustrates several Pythagorean triples. The smallest side length is an odd number.

a	b	c
3	4	5
5	12	13
7	24	25
9	40	41
11	60	61

Look for patterns in the table. For example, b and c are consecutive integers, b is even and c is odd. The sum $b + c$ appears to be a perfect square. Can you predict the triple in which the smallest number is 13? Can you predict a formula for generating any Pythagorean Triple with a, the smallest number, an odd number ≥ 3.

We can prove that, for n an odd integer ≥ 3, then $(n, \frac{n^2-1}{2}, \frac{n^2+1}{2})$ is a Pythagorean Triple. This proof will be left for the student.

If a triangle has two angles equal, then the two opposite sides are equal. That is, the triangle is isosceles. The proof of this is left for the student.



On the first night we proved the base angle theorem for isosceles triangles that states: if a triangle has two equal sides, then the two opposite angles are equal. The above statement is called a *converse*. When a statement and its converse are both true, we can state them together using *if and only if (IFF* for short). The *Isosceles Triangle Theorem* can be stated: A triangle has two equal sides IFF it has two equal angles.

If $\angle A < \angle B$, then $a < b$.

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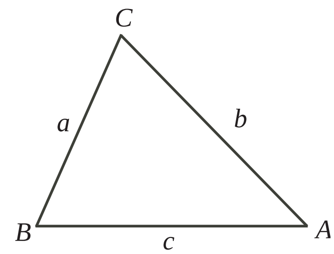
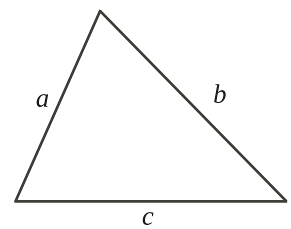
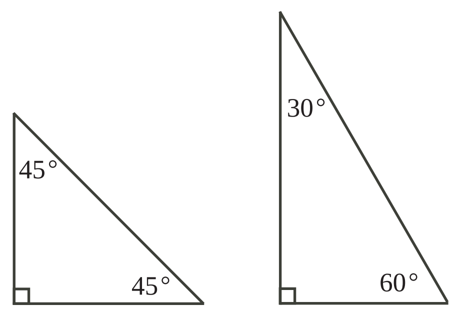


If a, b and c are the side lengths of a triangle, the *Triangle Inequality* tells us that $b + c > a$ and $a + c > b$ and $a + b > c$.
Can you explain why this is true?



There are two kinds of special triangles.
The first has angles 45°, 45° and 90°.
The second has angles 30°, 60° and 90°.
If the shortest side in each has length 1, what are the other side lengths?
These can be scaled by any factor.



Congruent Triangles

Two triangles are called *congruent* if corresponding side lengths and corresponding angles are all equal. In other words, the triangles are equal in all respects. Sometimes, fewer than these 6 equalities are necessary to establish congruence. Some ways to determine that two triangles are congruent:

- Side-Side-Side (SSS)
- Side-Angle-Side (SAS)
- Angle-Side-Angle (ASA)
- Right Angle-Hypotenuse-Side (RHS)

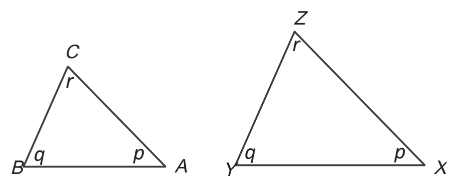
Once two triangles are proved to be congruent, all of the other corresponding equalities follow.

Similar Triangles

• Two triangles are called *similar* if corresponding angles are equal. If two triangles are similar, then the corresponding pairs of sides are in a constant ratio.

In this example, $\angle A = \angle X$ and $\angle B = \angle Y$ and $\angle C = \angle Z$. Therefore, $\triangle ABC \sim \triangle XYZ$.

Then $\dfrac{AB}{XY} = \dfrac{AC}{XZ} = \dfrac{BC}{YZ}$. In other words, the triangles are "scaled models" of each other.



• Two triangles are also similar if two pairs of corresponding sides are in constant ratio and the angles between the sides are equal.

Once similarity is shown then the corresponding pairs of sides are in a constant ratio.

In this example, $\dfrac{CB}{ZY} = \dfrac{CA}{ZX} = \dfrac{1}{3}$ and $\angle C = \angle Z$..

As a result of similarity, $\angle B = \angle Y$, $\angle A = \angle X$ and $\dfrac{BA}{YX} = \dfrac{1}{3}$.