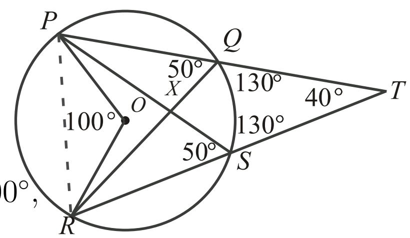
Intermediate Math Circles
Wednesday October 31 2012
Problem Set 4 Solutions

1. In the diagram, O is the centre of the circle. Determine the measure of $\angle QXS$.



Solution

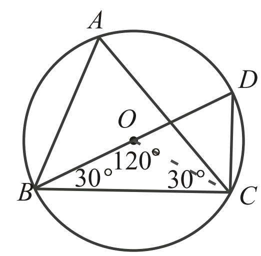
By the angle at the circumference property, $2\angle PQR = \angle POR = 100°$, so $\angle PQR = 50°$.

Inscribed angles $\angle PQR$ and $\angle PSR$ share a common chord PR, so they are equal and hence $\angle PSR = 50°$. As PT and RT are straight lines containing these angles respectively, it follows that $\angle XQT = \angle XST = 130°$.

In quadrilateral $QXST$,

$$\begin{aligned}\angle QXS + \angle XQT + \angle XST + \angle QTS &= 360° \\ \angle QXS + 130 + 130 + 40 &= 360 \\ \angle QXS + 300 &= 360 \\ \therefore \angle QXS &= 60°\end{aligned}$$

2. Determine the measure of $\angle BAC$.



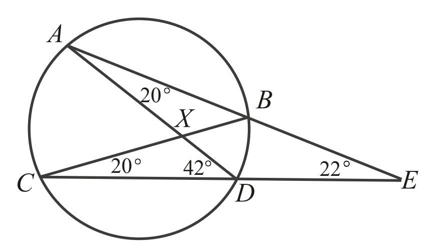
Solution

OB and OC are radii, so $OB = OC$ and $\triangle OBC$ is isosceles.
It follows that $\angle OCB = \angle OBC = 30°$, and $\angle BOC = 120°$.

By the angle at the circumference property, $2\angle BAC = \angle BOC = 120°$, so therefore $\angle BAC = 60°$.

(Note: By the angle inscribed by a common chord property, $\angle BDC = 60°$ as well, and hence $\angle DCB = 90°$. This implies BD is a diameter. You may have assumed that BD was a diameter in your original solution, and obtained the same result. However, this would be unnecessary, as you would have to prove that BOD is a straight line, and in the process, solved the answer anyway!)

3. Determine the measure of $\angle ADC$ and of $\angle AXB$.



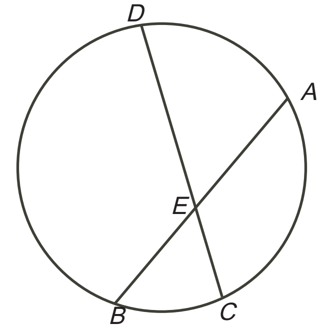
Solution

By the "inscribed angles in a circle by a common chord" property, $\angle DCB = \angle DAB = 20°$.

$\angle ADC$ is exterior to $\triangle DAE$, hence $\angle ADC = \angle DAE + \angle DEA$ $= 20 + 22 = 42°$.

Then in $\triangle CXD$, $\angle CXD = 180 - 20 - 42 = 118°$. It follows by the opposite angle theorem that $\angle AXB = \angle CXD = 118°$.

4. AB and CD are two intersecting chords in a circle.



a) If $AE = 6$, $BE = 4$ and $CE = 8$, determine the length of DE.

Solution

This is a straightforward application of the chord splitting property. Let $x = DE$.

DC and AB are chords which intersect at E, it follows that

$$\begin{aligned} DE \times CE &= AE \times BE \\ x \times 8 &= 6 \times 4 \\ x \times 8 &= 24 \\ x &= 3 \\ \therefore DE &= 3 \end{aligned}$$

b) If $AE = x$, $AB = 2x + 5$, $CE = x + 11$ and $CD = 2x + 7$, determine the value of x.

Solution

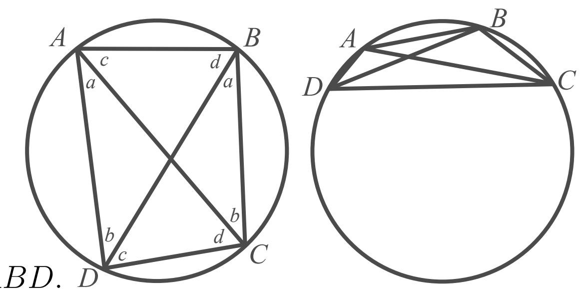
If $AE = x$ and $AB = 2x + 5$, then $BE = AB - AE = x + 5$.
Similarly, $DE = CD - CE = (2x + 7) - (x + 11) = x - 4$.

Apply the chord splitting property as before.

$$\begin{aligned} DE \times CE &= AE \times BE \\ (x - 4) \times (x + 11) &= x \times (x + 5) \\ x^2 - 4x + 11x - 44 &= x^2 + 5x \\ 7x - 44 &= 5x \\ 2x &= 44 \\ \therefore\ x &= 22 \end{aligned}$$

5. A *cyclic quadrilateral* is a quadrilateral that has all four of its vertices on the same circle. Prove that opposite angles are supplementary.

Solution



In the proof given, the leftmost circle will be used as reference. However, the proof is the same for any other configuration, such as the one found in the rightmost circle.

Let $a = \angle DAC$, $b = \angle ADB$, $c = \angle CAB$, $d = \angle ABD$.

Applying the "angle inscribed in a circle by a common chord" property,
$\angle DBC = \angle DAC = a$, $\angle ACB = \angle ADB = b$, $\angle CAB = \angle CDB = c$ and $\angle ABD = \angle ACD = d$. This has been labelled in the diagram.

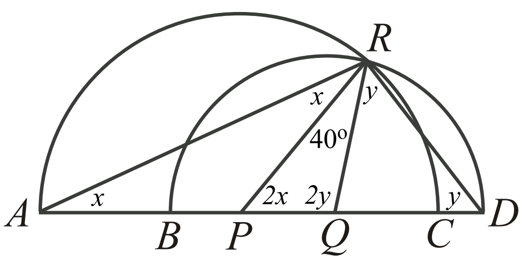
$ABCD$ is a quadrilateral; hence

$$\begin{aligned}\angle DAB + \angle ABC + \angle BCD + \angle CDA &= 360^\circ\\ (a+c)+(a+d)+(b+d)+(b+c) &= 360\\ 2a+2b+2c+2d &= 360\\ 2(a+b+c+d) &= 360\\ (a+b+c+d) &= 180^\circ\end{aligned}$$

Observe that $\angle DCB$ and $\angle DAB$ are opposite angles in the quadrilateral, and that $\angle DCB + \angle DAB = (b+d)+(a+d) = a+b+c+d = 180^\circ$. So these two are supplementary.

Similarly, it can be shown that $\angle ABC + \angle CDA = 180^\circ$, and hence these two opposite angles are also supplementary. Therefore, in a cyclic quadrilateral, opposite angles are supplementary.

6. In the diagram, points B, P, Q, and C lie on line segment AD. The semi-circle with diameter AC has centre P and the semi-circle with diameter BD has centre Q. The two semi-circles intersect at R. If $\angle PRQ = 40^\circ$, determine the measure of $\angle ARD$.



Solution

In the semi-circle with diameter AC, PR and PA are radii. It follows that $\triangle RPA$ is isosceles, and hence $\angle PRA = \angle PAR = x$. Note that $\angle RPC$ is exterior to $\triangle RPA$; then $\angle RPC = \angle PRA + \angle PAR = 2x$.

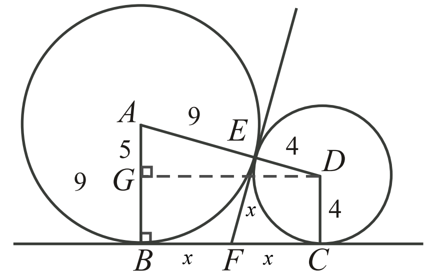
Similarly, in the semi-circle with diameter BD, QR and QD are radii, so $\triangle QRD$ is isosceles. It follows $\angle QRD = \angle QDR = y$. Also, $\angle RQB$ is exterior to $\triangle QRD$, so $\angle RQB = 2y$.

Then in $\triangle PRQ$,

$$\begin{aligned}180^\circ &= \angle RPQ + \angle RQP + \angle PRQ\\ 180 &= 2x + 2y + 40\\ 140 &= 2(x+y)\\ 70^\circ &= x + y\end{aligned}$$

Observe that $\angle ARD = x + y + 40$. Therefore, by substituting the above result, $\angle ARD = (x+y) + 40 = 70 + 40 = 110^\circ$.

7. In the diagram, a circle with centre A and radius 9 is tangent to a smaller circle with centre D and radius 4. Common tangents EF and BC are drawn to the circles making points of contact at E, B, and C. Determine the length of EF. (For this question you may have to use properties which make sense but are, as of yet, unproven.)



Solution

Two circle properties will be used without proof in this solution:

(1) A tangent line to a circle is perpendicular to the radius of the circle which passes through the point of tangency.

(2) If two tangents to a circle, when extended, meet at a common point, the distance from either point of tangency to the intersection point is the same.

From (1), it can be concluded that AD is a straight line. From (2), $BF = EF = FC = x$, as has been labelled in the diagram.

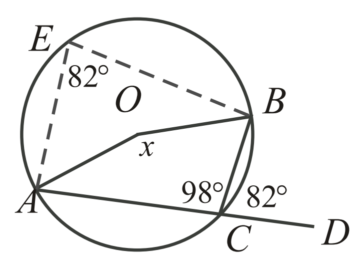
Since DE and DC are radii of the same circle, $DE = DC = 4$. Similarly, $AE = AB = 9$. Since AD is a straight line, then $AD = AE + DE = 13$.

DG has been constructed parallel to BC. Since BC was perpendicular with AB, by the interior angles property of parallel lines, $\angle BGD = \angle AGD = 90°$. So $\triangle AGD$ is right angled. It follows that $GB = DC = 4$. Then $AG = AB - GB = 9 - 4 = 5$. Applying the Pythagorean Theorem to $\triangle AGD$,

$$\begin{aligned} AG^2 + GD^2 &= AD^2 \\ 5^2 + GD^2 &= 13^2 \\ GD^2 &= 144 \\ GD &= 12 \qquad (GD > 0) \end{aligned}$$

Observe that $GDCB$ is a rectangle; hence $BC = GD = 12$. Since $BC = BF + FC = 2x$, $x = 6$. Therefore, $EF = x = 6$.

8. If O is the centre of the circle and $\angle BCD = 82°$, what is the value of x in degrees?



Solution

AD is a straight line; $\angle BCD = 82°$ and hence $\angle BCA = 98°$.
Let E be any point on the arc between A and B.
This forms an inscribed (cyclic) quadrilateral $EACB$.

From problem 5, it was shown that opposite angles in a cyclic quadrilateral are supplementary. Then $\angle AEB = 180 - \angle BCA = 180 - 98 = 82°$.

Applying the angle at the circumference property, $2\angle AEB = \angle AOB = x$, and hence $x = 2(82) = 164°$.

(Within the solution, an interesting result was proven - given a cyclic quadrilateral, any exterior angle is equal to its opposite interior angle within the quadrilateral.)