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# Grade 6 Math Circles
# Binary Numbers
November 8, 2012

**NOTE:** The Wednesday lesson was modified from the Tuesday lesson to be more suitable for a Grade 6 level. For those in the Tuesday class, please refer to these (Wednesday) materials in the future. The solutions posted are for the problems in the Wednesday class. If you want solutions to the Tuesday problem, send me an email at ttzhou@uwaterloo.ca

The number system you are all familiar with is BASE 10. In this number system, we use the digits `0 - 9`.

Today we are going to be learning about a new type of number system - binary.

To help us understand this number system, we need to first learn a little about exponent notation.

## Exponent Notation

When we write $5 \times 6$, we really mean "$5 + 5 + 5 + 5 + 5 + 5$" (6 times). Multiplication $\rightarrow$ repeated addition.

Exponent notation $\rightarrow$ repeated multiplication.

$$\underbrace{5 \times 5 \times 5 \times 5 \times 5 \times 5}_{\text{6 times}} = 5^6$$

$$2^4 = \underbrace{2 \times 2 \times 2 \times 2}_{\text{4 times}}$$

For example, when we want to say "4 multiplied by itself 6 times", we say "4 to the EXPONENT of 6", and we write

$$\underbrace{4^6 = \underline{\hspace{3cm}}\overline{\hspace{3cm}}}_{\texttt{POWER}}$$

**Examples**

1. $1000 = 10 \times 10 \times 10 = 10^3$
2. $256 = 2\times2\times2\times2\times2\times2\times2\times2 = 2^8$

**Note**: Any number (except 0) to the exponent of 0 equals 1 - $5^0, 12314134134^0$ all equal 1!

**Exponents and Base 10**

Let's look at how Base 10 really works... Consider the number 904. What this number actually represents is

| Place | 100 | 10 | 1 |
|---|---|---|---|
| Digit | 9 | 0 | 4 |

| Place | $10^2$ | $10^1$ | $10^0$ |
|---|---|---|---|
| Digit | 9 | 0 | 4 |

Another way to write this out is as a SUM

$$(9 \times 100) + (0 \times 10) + (4 \times 1) = \mathbf{904}$$
$$(9 \times 10^2) + (0 \times 10^1) + (4 \times 1) = \mathbf{904}$$

**Practice** - On a separate sheet of paper, try writing out the following numbers in a place value table (using exponents), and also as a sum like above (using exponents)

1. 111
2. 80
3. 1350

## Translating from one to another

In the Base 10 table, each column was 10 times greater than the one to its right. This is why our number system is called `BASE 10`, or the DECIMAL system (the powers in the columns have a BASE of 10).

BINARY is the name we give to the `BASE 2` system.

Let's first learn how to convert from Base 2 to Base 10. In Base 2, each column is TWICE greater than the one to its right.

In base ten, each column could have a digit from 0 to 9. In binary/base 2, each column can have a digit only from 0 to 1.

Let's convert a binary number, 10011, back into Base 10.

1. Count how many digits you have. In our case, we have 5.
2. Make the table like below with 5 columns, and fill in the places in the "Place" row, from left to right, starting with $2^4$, then $2^3$, and so on to the right until you hit the last column, which will have $2^0$.

| Place | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ |
|---|---|---|---|---|---|
| Place | 16 | 8 | 4 | 2 | 1 |
| Digit | 1 | 0 | 0 | 1 | 1 |

3. Fill in the digits column underneath the places with either a 0 or 1.
4. Now, reading the "Place" row from left to right, add up all the numbers that have a 1 below it in the "Digit" row. Your sum will be the base 10 conversion.
5. For example, using 10011...

$$(1 \times 2^4) + (1 \times 2^1) + (1 \times 2^0) = ?$$
$$16 + 2 + 1 = 19$$

**Practice** - Convert the following binary numbers into their base 10 equivalent (use another sheet of paper to do these).

1. 1111 $\rightarrow$ 15
2. 1110 $\rightarrow$ 14
3. 10110 $\rightarrow$ 22

Now let's see how to go from Base 10 to binary! We are going to convert 116 (base 10) into a binary number.

1. Find the highest power of 2 that is less than or equal to the number. In our case, it is $64 = 2^6$.
2. Create the table shown below, where the leftmost column in the "Place" row is the number you got in step 1.

| Place | $2^6$ | $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ |
|---|---|---|---|---|---|---|---|
| Place | 64 | 32 | 16 | 8 | 4 | 2 | 1 |
| Digit | | | | | | | |

3. Fill in the leftmost column (here it is $2^6$) with a 1 in the "Digit" row. Keep a running total. Subtract the place value of this column from your number. In this case, running total is $116 - 64 = 52$.
4. Using your new total, repeat Step 1. Underneath that place value, put a 1. In this case, it is $2^5 = 32$. Then subtract the place value of that column from your running total. So in this case, $52 - 32 = 20$.
5. Repeat the process until there are no more columns, always keeping track of your running total. When you are done, fill in the remaining columns with a 0.

| Place | $2^6$ | $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | 1 |
|---|---|---|---|---|---|---|---|
| Place | 64 | 32 | 16 | 8 | 4 | 2 | 1 |
| Digit | 1 | 1 | 1 | 0 | 1 | 0 | 0 |

**Practice** - Try converting the following numbers into their binary form.

1. 16 $\rightarrow$ 10000
2. 56 $\rightarrow$ 11100
3. 65 $\rightarrow$ 100001

**Problems**

For the following, those marked with a single "*" require a little more thinking than the others... Those with more than one "*" are more difficult and are meant to be challenge questions.

1. What number does $(1 \times 2^3) + (0 \times 2^2) + (0 \times 2) + (1 \times 1)$ represent in `BASE 2`?
2. What number does $(1 \times 10^5) + (8 \times 1)$ represent in `BASE 10`?
3. What number does $(1 \times 2^5) + (1 \times 2) + (1 \times 1)$ represent in `BASE 10`?
4. What number does $(8 \times 2^2)$ represent in `BASE 2`?
5. Convert the following `BASE 10` numbers into their binary equivalent.
   (a) `14`
   (b) `6`
   (c) `64`
   (d) `122`
6. Convert the following `BASE 2` numbers into their decimal (`BASE 10`) equivalent.
   (a) `111`
   (b) `1001`
   (c) `110001`
   (d) `10`
7. * Why can there be no `BASE 1` number system?
8. Perform the following additions in binary by converting them into base 10 numbers first, and then converting the answer back.
   (a) `1 + 1 =`
   (b) `1 + 10 =`
   (c) `111 + 10 =`
   (d) `10000 + 10101 =`

9. In World War II, some of the most important, secret jobs were given to the cryptographers (codebreakers)! See if you can crack the following encoded quotes by converting the binary numbers into `BASE 10`, and then matching the `BASE 10` number to the corresponding alphabet letter (for example, `20` $\rightarrow$ `T`, `1` $\rightarrow$ `A`, `6` $\rightarrow$ `F`).

    ```
    111-1001-10110-101   1101-101   1   10000-1100-1-11-101
    10100-1111   10011-10100-1-1110-100   1111-1110   1-1110-100
    1001   10111-1001-1100-1100   1101-1111-10110-101
    10100-1000-101   101-1-10010-10100-1000.
    ```

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    ```
    1001   10100-1000-1001-1110-1011,
    10100-1000-101-10010-101-110-1111-10010-101   1001   1-1101
    ```

    - RENE DESCARTES

10. * Convert the following `BASE 10` numbers into their `BASE 8` equivalent.

    (a) 8

    (b) 64

    (c) 67

    (d) 88

11. ** (Challenge)

    (a) How many possible binary numbers with 5 digits are there?

    (b) How many possible 5 digit binary numbers are there where the last digit must be a 0?

    (c) How many are there where the second and last digits must be a 0?

12. *** (Challenge) For a given length $n \geq 2$

    (a) How many possible binary numbers with $n$ digits are there?

    (b) How many possible binary numbers are there where the last digit must be a 0?

    (c) How many are there where the second and last digits must be a 0 (assume $n \geq 3$ )?