UNIVERSITY OF WATERLOO

University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

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Probability

Probability is one of the most prominent uses of mathematics in our everyday lives. Knowing the probability of a certain event happening or not happening can be very important to us in the real world. The following are examples of situations that take place in our everyday lives that involve the use of probability:

- Weather Forecasting
- Lottery
- Sports: batting averages, free throw percentage, field goal percentage
- Medical decisions: operation success rates

Basic Probability

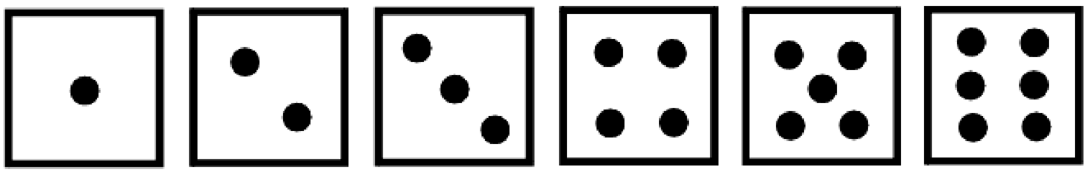
When calculating probability, we use the following formula:

$$P(A) = \frac{\text{\# of favourable outcomes to event A}}{\text{total \# of outcomes}}$$

Before we can use this formula, we need to understand a few things.

Definition: the sample space refers to the complete set of all possible outcomes.

Example 1: All possible outcomes of rolling a six-sided regular die are: 1, 2, 3, 4, 5, 6.



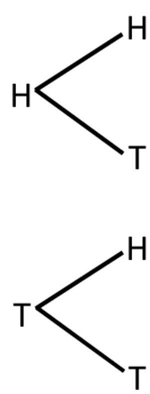
We can usually represent the sample space using a tree diagram or a chart.

Example 2: A regular coin is flipped three times. Using a tree diagram, illustrate the sample space.

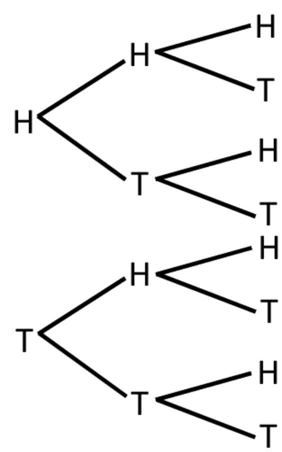
Solution:

1. Start the tree diagram by indicating the two possible results from the first flip.

2. Continue the tree diagram by connecting all possibilities for the second flip.



3. Complete the tree diagram by connecting all possibilities for the third flip.



Therefore the sample space is: **HHH HHT HTH HTT THH THT TTH TTT** and so there are 8 possible outcomes.

Exercise 1: Two dice are rolled, then the sum of the dice is recorded. Illustrate the same space in a chart.

Dice 1 \ Dice 2	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

The sample space for the sum of two dice is: $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Now, answer the following. How many possible outcomes have a sum of:

a) 3? 2

b) 7? 6

c) 9? 4

d) 12? 1

Recall: *Fundamental Counting Principle*

> If there are a ways that one event can occur and b ways that another event can occur, then both events can occur in $a \times b$ ways.

So, let's consider example 2 again. The first flip has two possible outcomes, heads or tails. The second flip has two possible outcomes as well, heads or tails. The third flip also has two possible outcomes, heads or tails.

Using the Fundamental Counting Principle, we can calculate the size of our sample space: $2 \times 2 \times 2 = 8$ total possible outcomes.

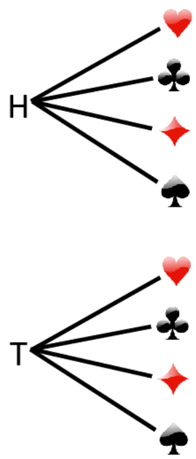
Example 3: A regular coin is flipped and then a card is randomly drawn from a standard deck of 52 cards.

a) Determine the probability of flipping a head, then drawing a diamond.

b) Determine the probability of flipping a head, then drawing a diamond or a heart.

Solution:

a) Step 1: Begin by drawing the sample space.



Step 2: Now apply the formula. There is only 1 favourable outcome our of 8 possible outcomes.

$$P(\text{Head, Diamond}) = \frac{1}{8}$$

So the probability that we will flip a head, then drawing a diamond is 1/8.

b) Using the sample space above, there are two favourable outcomes out of eight.

$$P(\text{Head, Diamond } \textbf{OR} \text{ Head, Heart}) = \frac{\text{Favourable Cases}}{\text{Total Cases}} = \frac{2}{8} = \frac{1}{4}$$

Adding Probabilities

The probability of event A **OR** event B happening, if the events are mutually exclusive, may be found by adding the individual probabilities.

$$P(\text{A } \textbf{OR} \text{ B}) = P(\text{A}) + P(\text{B})$$

Definition: The phrase mutually exclusive means that one event taking place prevents the other event from taking place.

For example, flipping a coin gives mutually exclusive outcomes since you can't get heads and tails at the same time. Another example, tossing a six-sided die are mutually exclusive since no two results can occur at the same time.

An example of events that are not mutually exclusive is choosing a King and choosing a heart, since you can choose the King of hearts from a standard deck.

Example 4: Two fair six-sided dice are rolled and the sum is recorded. If the probability of rolling a sum greater than 10 is 0.083, and the probability of rolling a sum less than 5 is 0.167, what is the probability of rolling a sum greater than 10 or a sum less than 5?

Solution: Let event A be rolling a sum greater than 10. Let event B be rolling a sum less than 5.

$$P(\text{A}) = 0.083 \text{ and } P(\text{B}) = 0.167$$

Since the two events are mutually exclusive (ie you cannot have a sum less than 5 and greater than 10 at the same time), then the probability of event A or event B happening is:

$$P(\text{A } \textbf{OR} \text{ B}) = P(\text{A}) + P(\text{B}) = 0.083 + 0.167 = 0.25$$

Multiplying Probabilities

The probability of event A **AND** event B happening, if the events are independent, may be found by multiplying the individual probabilities.

$$P(\text{A } \textbf{AND} \text{ B}) = P(\text{A}) \times P(\text{B})$$

Definition: Two events are independent if the outcome of one event does not change the probability of the second event occuring.

For example, a student tosses a fair six-sided die and gets a 2, then tosses the die again and gets a 6. The two events do not influence each other in any way, so they are independent.

Two events that not independent (or dependent) is if you take a marble out of a bag and without replacing that marble, you take another marble out of the bag.

Example 5: A regular coin is flipped, then a card is randomly chosen from a standard deck of 52 cards. What is the probability of flipping a head and choosing a diamond?

Solution: Let event A be flipping a head. Let event B be drawing a diamond.

$$P(\text{A}) = \tfrac{1}{2} = 0.5$$

$$P(\text{B}) = \tfrac{13}{52} = \tfrac{1}{4} = 0.25$$

$$P(\text{A } \textbf{AND} \text{ B}) = P(\text{A}) \times P(\text{B}) = 0.5 \times 0.25 = 0.125 = \tfrac{1}{8}$$

Complement

The probabilities of mutually exclusive events add up to one.

For example, let's consider a coin. The probability of getting a head is 0.5, and getting a tail is 0.5. There is a 100% chance of getting a head *or* a tail when you flip a coin, so the total probability is 1.

Suppose you now have a weighted coin, where the probability of getting a head is 0.43.

To calculate the probability of getting a tail, just subtract the given probability from 1.

$$P(\text{tail}) = 1 - P(\text{head}) = 1 - 0.43 = 0.57$$

This is called the complement probability.

Another way of representing the complement for an event A, is by using the following expression.

$$\bar{\text{A}}$$

The dash above the A tells us to calculate the probability that event A does not happen.

Example 6: If $P(\text{A}) = 0.25$, calculate $P(\bar{\text{A}})$.

Solution:

$$\begin{aligned} P(\bar{\text{A}}) &= 1 - P(\text{A}) \\ &= 1 - 0.25 \\ &= 0.75 \end{aligned}$$

Exercise 2: A weighted coin is altered so the probability of it landing on a head for each flip is $\frac{5}{7}$. The trick coin is flipped 3 times. What is the probability of getting a tail on the first flip and heads on the next two flips?

Solution:

Let event A be flipping a head.

$P(\text{A}) = \frac{5}{7}, \quad P(\bar{\text{A}}) = 1 - P(\text{A}) = 1 - \frac{5}{7} = \frac{2}{7}$

Since the events are independent, we can use the multiplying probabilities formula.

$P(\bar{\text{A}} \text{ AND A AND A}) = P(\bar{\text{A}}) \times P(\text{A}) \times P(\text{A}) = \frac{2}{7} \times \frac{5}{7} \times \frac{5}{7}$

Problem Set

1. What is the sample space for choosing a letter from the word "MATHEMATICS"?

2. What is the sample space for rolling a fair six-sided die and flipping a fair coin?

3. What is the probability of choosing a 5 from a standard deck of playing cards?

4. From a jar which contains 3 red, 5 green and 7 blue marbles, what is the probability of:

   (a) choosing a blue marble from a jar?

   (b) choosing a marble that is not red?

   (c) choosing a red or a green marble?

5. When rolling a regular six-sided die, what is the probability of:

   (a) rolling an even number?

   (b) rolling an odd number?

6. What is the probability of landing on an odd number after spinning a spinner with 7 equal sections numbered 1 through 7?

7. What is the probability of choosing a face card when drawing a card from a standard deck of 52 playing cards?

8. If a number is chosen at random from the numbers 1 to 20 inclusive, what is the probability that:

   (a) a prime number will be picked?

   (b) a single digit number will be picked?

   (c) a number containing the digit 1 will be picked?

9. When rolling two fair dice, what is the probability that the sum is:

   (a) greater than 5?

   (b) less than 2?

   (c) equal to 8?

   (d) less than or equal to 12?

   (e) greater than or equal to 4?

10. When choosing two cards from a standard deck of 52, what is the probability of (with replacement):

    (a) choosing the Jack of Diamonds on the first draw and the 5 of Clubs on the second draw?

    (b) choosing a Jack on the first draw and a 5 on the second draw?

    (c) choosing a 10 on the first draw and the 3 of Spades on the second draw?

    (d) choosing the 8 of Hearts on the first draw and the 8 of Hearts on the second draw?

    (e) choosing any card on the first draw and choosing the same card on the second draw?

11. When rolling a fair six-sided die twice, what is the probability of:

    (a) rolling an even number and then rolling a number greater than 3?

    (b) rolling a 6 and then rolling a 2?

    (c) rolling any number and then rolling an odd number?

12. ** In the city of Waterloo, the probability that it will snow is 35%. What is the probability that:

    (a) it will snow at least 3 out of the next 5 days? (Answer: 0.43)

    (b) it will snow exactly 3 out of the next 5 days? (Answer 0.18)

    (c) it will snow at most 2 out of the next 5 days? (Answer: 0.57)

13. *** In Canada, the probability that someone plays baseball and/or hockey is 0.79. The probability that someone plays just hockey is 0.6 and the probability that some plays baseball and hockey is 0.15. What is the probability that some plays only baseball? (Answer: 0.34)