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Grade 7/8 Math Circles
Sequences and Series
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What are sequences?

A **sequence** is an **ordered** list of items (whole numbers, fractions, names, etc.) Usually, the order of the elements follows some **rule** or pattern. We highlight the terms in the sequence using curly braces like these: { }

For example, $\{2, 4, 6, 8 \dots\}$ is a sequence whose elements increase by **two** each time.

An element in the sequence is called a "**term**". The twentieth term in that sequence is 40. The rule for the n^{th} term (we call this t_n) of this sequence is

$$t_n = 2n$$

Checkpoint: What is t_{1750} in this sequence?

$$t_{1750} = 2(1750) = 3500$$

Describing Sequences

Mathematicians by nature are lazy. We hate writing out long sequences; instead, we write short, precise ways to describe them.

If we can express t_n in terms of **only** n, we say we have a **closed form** expression.

$t_n = 2n$ is an example of a closed form expression that describes the sequence. So we would describe $\{2, 4, 6, 8, \dots\}$ as $2n$ instead.

Another way we can describe the sequence is to define the n^{th} term using the terms that come before it.

For example, in our sequence, notice that **current term = previous term** + 2. Mathematically, this is $t_n = t_{n-1} + 2$. t_n always stands for the current term. t_{n-1} always stands for the previous term.

When we express t_n in terms of previous terms in the sequence, we are using a **recursive** expression.

Did we catch everything in our description?

Can you see any problems with only using $t_n = t_{n-1} + 2$ to describe the pattern?

Lacking a STARTING POINT!

Recursive sequences require both the rule AND a starting point - we call this the **initial term**. In general, we usually describe a sequence in one of the two above ways.

Let's take a look at some more sequences:

1. $\{3, 5, 7, 9, 11, 13, 15, \ldots\}$
   (a) Recursive: $t_n = t_{n-1} + 2$
   (b) Initial Term: 3
   (c) Closed Form: $t_n = 2n + 1$

2. $\{2, 4, 8, 16, 32, 64, 128, 256, \ldots\}$
   (a) Recursive: $t_n = 2 \times t_{n-1}$
   (b) Initial Term: 2
   (c) Closed Form (if you've seen exponents before): 2^n

3. * $\{2, 3, 5, 9, 17, 33, 65, \ldots\}$
   (a) Recursive: $t_n = 2(t_{n-1}) - 1$
   (b) Initial Term: 2

4. $\{1, 4, 9, 16, 25, 36, 49, 64, \ldots\}$
   (a) Closed Form: $t_n = n^2$

Arithmetic Sequences

Notice that in the first sequence there is a common **difference** between a term and the term that follows it. Indeed,

$$5 - 3 = 13$$
$$7 - 5 = 13$$
$$\vdots$$
$$9 - 7 = 13$$
$$\vdots$$

Whenever there is a **common difference** between a term in a sequence and the one immediately before or after it, we call it an **arithmetic sequence.**

- The 2nd term (5) is 2 greater than the first (3).
  - So $t_2 = 5 = 3 + 2 = 3 + (1)(2)$.
- The 3rd term (7) is 2 greater than the second, and so is $2 + 2 = (2)(2) = 4$ greater than the first.
  - So $t_3 = 7 = 5 + 2 = (3 + 2) + 2 = 3 + 2 + 2 = 3 + (2)(2)$.
- The 4th term (9) is 2 greater than the third, and so is $2 + 2 + 2 = (3)(2) = 6$ greater than the first.
  - So $t_4 = 9 = 7 + 2 = 5 + 2 + 2 = 3 + 2 + 2 + 2 = 3 + (3)(2)$.
- Can you see the pattern? How much greater is the n^{th} term than the first?

The general formula for an arithmetic sequence is then

$$\boxed{n^{th}\text{ term}(t_n) = (\text{first term}) + (n - 1)(\text{common difference})}$$

- **First term** is easy to find... although it's pretty hard for me!
- **"Common Difference"** is easy to find as well!
  - Pick any term after the first, and then subtract it by its previous term
  - $\{11, 25, 39, 53, 67, \ldots\}$ - pick 67, subtract 53. So 14 is the common difference! Make sure you watch the order you subtract the terms!

Checkpoint: Can you write a mathematical expression for the first sequence?

$$7 + (n - 1)(13)$$

Checkpoint: Can you find the 46^{th} term in the sequence?

$$t_{46} = 7 + (46 - 1)(13) = 592$$

Three more examples...

1. Find the arithmetic sequence formula for the sequence $\{8, 13, 18, 23, 28, 33, \ldots\}$
   - First Term: 8
   - Common Difference: 5
   - Arithmetic Sequence Formula: $t_n = 8 + (n - 1)(5)$

2. Find the arithmetic sequence formula for the sequence $\{2, -3, -8, -13, -18, -23 \ldots\}$
   - First Term: 2
   - Common Difference: -5
   - Arithmetic Sequence Formula: $t_n = 2 + (n - 1)(-5)$

3. Find the arithmetic sequence formula for the sequence $\{-12, -3, 6, 15, 24, 33 \ldots\}$
   - First Term: -12
   - Common Difference: 9
   - Arithmetic Sequence Formula: $t_n = -12 + (n - 1)(9)$

Practice, Practice, Practice!!!

For each of the following sequences, describe the sequence, and write out its general arithmetic sequence formula. Then use the formula to find the 12^{th} term in the sequence.

1. $\{-11, -2, 7, 16, 25, 34, \ldots\}$
   - Recursive: $t_n = t_{n-1} + 9$
   - Arithmetic: $t_n = -11 + (n-1)(9)$
   - Closed Form: $t_n = 9n - 20$
   - 12th term is, using the arithmetic formula, -11 + (12 - 1)(9) = 88.

2. $\{-13, -15, -17, -19, -21, -23, \ldots\}$
   - Recursive: $t_n = t_{n-1} + (-2)$
   - Arithmetic: $t_n = -13 + (n-1)(-2)$
   - Closed Form: $t_n = 11 - 2n$
   - 12th term is -35.

3. $\{15, 4, -7, -18, \ldots\}$
   - Recursive: $t_n = t_{n-1} + (-11)$
   - Arithmetic: $t_n = 15 + (n-1)(-11)$
   - Closed Form: $t_n = 26 - 11n$
   - 12th term is -106

4. $\{1, 1.5, 2, 2.5, 3, 3.5, \ldots\}$
   - Recursive: $t_n = t_{n-1} + (0.5)$
   - Arithmetic: $t_n = 1 + (n-1)(0.5)$
   - 12th term is $\frac{13}{2}$

Finding the Number of Terms in a FINITE Arithmetic Sequence

Finite arithmetic sequences are arithmetic sequences where there is an end. Most of the ones we've seen so far have been neverending, but that's not always the case.

We saw that $\{1, 3, 5, 7, \dots\}$ was an INFINITE sequence because the $\dots$ meant that the sequence goes on forever.

The sequence $\{1, 3, 5, 7, \dots, 155\}$ DOES stop - it stops at 155. This means it is FINITE - FINITE means that the sequence eventually stops.

How can we find how many terms are in this arithmetic sequence, or any FINITE arithmetic sequence, where we know the last term?

Well, we know that we have an arithmetic sequence, and we know each term is 2 more than the one before it. We can use this to figure out how many terms - but how?

Let's first find the arithmetic sequence formula for this sequence.

- First Term: 1
- Common Difference: 2
- Formula: $t_n = 1 + (n - 1)(2)$

We want to find n when $t_n = 155$ - so let's plug in 155 for t_n, and solve for n!

$$\begin{aligned}
155 &= 1 + (n - 1)(2) \\
155 - 1 &= 1 + (n - 1)(2) - 1 \\
154 &= (n - 1)(2) \\
154 \div 2 &= (n - 1)(2) \div 2 \\
77 &= (n - 1) \\
78 &= n
\end{aligned}$$

This tells us that 155 is the 78^{th} term in the sequence. Therefore, there must be 78 terms in the entire sequence!

Number of Terms = (Last Term - First Term) ÷ (Common Difference) + 1.

Problem Set

Describe the following sequences, using either a recursive definition or closed form expression. If possible, find the arithmetic sequence general formula. Then find the...

1. $\underline{10^{th}\text{ term}}$ of $\{3, 6, 9, 12, 15, 18, \dots\}$
2. $\underline{8^{th}\text{ term}}$ of $\{3, 11, 19, 27, 35, \dots\}$
3. $\underline{11^{th}\text{ term}}$ of $\{25, 23, 21, 19, 17, \dots\}$
4. $\underline{11^{th}\text{ term}}$ of $\{125, 120, 115, 110, 105, \dots\}$
5. $\underline{21^{st}\text{ term}}$ of $\{-25, -23, -21, -19, -17, \dots\}$
6. $\underline{6^{th}\text{ term}}$ of $\{1, 4, 16, 64, 256, \dots\}$
7. * $\underline{13^{th}\text{ term}}$ of $\{1, 3, 7, 15, 31, 63, 127, 255, \dots\}$
8. $\underline{26^{th}\text{ term}}$ of $\{0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, \dots\}$ (This one doesn't have a closed form or recursive expression - just use logic!)
9. $\underline{26^{th}\text{ term}}$ of $\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots\}$ (This one doesn't have a closed form or recursive expression - just follow the pattern!)
10. How many terms are in the sequence $\{1, 3, 5, 7, \dots, 123\}$?
11. How many terms are in the sequence $\{1, 5, 9, 13, \dots, 121\}$?
12. The sum of the interior angles of a triangle is 180, of a quadrilateral is 360 and of a pentagon is 540. Describe the pattern, and use it to find the sum of the interior angles of a dodecagon (12 sides). If the dodecagon is regular, what is the measure of one interior angle?
13. ** Find the $\underline{11^{th}\text{ term}}$ of $\{0, 6, 24, 60, 120, 210, 336, 504, 720, \dots\}$. Hint - think about the sequence with the closed form expression n^3. (Don't feel discouraged - I am not expecting you to get this one without a lot of work!)
14. ** Ones digit of the $\underline{63^{rd}\text{ term}}$ of $\{7, 7^2, 7^3, 7^4, 7^5, 7^6, \dots\}$ (Try actually evaluating a few of the terms in the sequence, and then looking at the ones digit of each of the terms. Do you notice a pattern? Think about question 8.)