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# Grade 7/8 Math Circles
# Sets and Venn Diagrams
October 26, 2012

## What is a Set?

Mathematics, at its very core, can be described ENTIRELY in terms of **sets**. But what is a set?

Simple - a set is just a collection of objects. These objects don't have to be numbers; they can be **anything**... people, planes, Saturday morning cartoons, etc. Items that belong to the set are called **elements** of the set.

We use curly braces ("{" and "}") to represent a set. Examples of sets include:

- {Mr. Zhou's Favourite Superheroes}
- {Your Favourite Singers}
- {The Positive Integers} (sets can be INFINITELY large)

There are two main ways to describe a set. You can use a general description, like we did above, or you can list out all the elements individually, like below:

- {Batman, Dr. Manhattan, Thor, The Hulk, Wolverine}
- {Justin Bieber, Miley Cyrus}
- $\{1, 2, 3, 4, 5, 6, 7, \dots\} = \{2, 4, 1, 3, 5, 6, 7, \dots\}$ (order doesn't matter)

We often just represent a set with a capital letter; that way, when we want to refer to the set again, we just use the letter.

- S = {Batman, Dr. Manhattan, Thor, The Hulk, Wolverine}
- L = {Justin Bieber, Miley Cyrus}
- $N = \{1, 2, 3, 4, 5, 6, 7, \dots\}$

When an element belongs to a set, we say that it is a **member** of the set, and we use the symbol $\in$ to denote this. For example, we'd write "Justin Bieber" $\in$ L.

## Working with Sets

Just like when we perform arithmetic **operations** (plus, minus, times) with numbers to get new numbers, we can perform certain operations on sets to get new sets. We will look at three main actions between sets: **union, intersection, complement**.

## The Union of Two Sets

Let's say two parties were booked separately, but not enough people showed up to either, so the party planner decided to combine them into one.

Let A = {Jimi, Kurt, Chris, Paul}

Let B = {Ronnie, Eddie, Axel, Steven, Mick, Jimi, Robert}

Then C, the set of all people who are going to show up at the new party, will be

C = {Jimi, Kurt, Chris, Paul, Ronnie, Eddie, Axel, Steven, Mick, Robert}

C is the list of people who were in Set A **or** in Set B. Note that "or" in this case really means "*in one or both*". C is thus known as the **union** of A and B. We write this as $C = A \cup B$.

Note that Jimi was in both A and B originally, but we only write him once in C. This is because **a set cannot contain duplicates**. If the union of two sets contains two elements that are the same, we only write that element once. When we do the union, we lump together both sets, and remove duplicates.

We can use Venn Diagrams to help us visualize what the union of two sets is. Shade everything in A and everything in B. Everything that is shaded will be a part of $A \cup B$, even if A and B overlap.

### The Intersection of Two Sets

The Disco Club is looking for a treasurer. They want someone experienced with money, and whom is also capable of working nights.

Let M = {Rocky, Apollo, Ivan, Adrian}

Let N = {Mick, Paulie, Apollo, Ivan, Adrian}

Then O, the set of capable candidates, will be

$$O = \{\text{Apollo, Ivan, Adrian}\}$$

O is the list of people who were in Set M **and** in Set N. O is called the **intersection** of M and N. We write this as $O = M \cap N$. Basically, the intersection contains the elements that both sets have in common.

Let's look at another situation. Say we were working with the same group of people, but this time we wanted to find a student council president. The candidate would need to have good leadership skills, but also have good dance skills.

Let K = {Apollo, Ivan}

Let L = {Mick, Adrian, Rocky}

Then the list of candidates would be P = {}. That is, there are no valid candidates - the intersection set is empty! We give this set a special name - the **empty set**, and we write it as either "{}" or represent it with the symbol, $\phi$.

The empty set is very special because the intersection of a set and the empty set is itself an empty set, while the union of a set and the empty set is the just the original (non-empty) set! Can you explain why?

Let's look at how intersections work visually, using a Venn Diagram again.

**Checkup**

Find the union and intersection of the two sets:

1. A = {1, 2, 3, 4, 5}; B = {1, 2, 9}

2. M = {PS3, Xbox, Wii}; N = {Wii, Xbox, PS3, PSP, DS}

3. S = {Tiger, Lion, Leopard, Cougar}; T = {Wolf, Hyena, Rottweiler}

4. X = $\phi$ = {}; Y = {Euclid, Archimedes, Pappus, Thales, Pythagoras}

**Subsets and the Universal Set**

In our previous example, notice that all of the elements in S were also in T. In this case, we say that S is **contained** in T, or that S is a **subset** of T. We write this as $S \subseteq T$. In the Disco Club example, neither was contained in the other, so we cannot write that one is a subset of the other.

When we work with sets, we often have to define something called the **universal set** - this is the set of all things we are interested in. All of the sets that we work with must have elements in this universe - therefore, each set we work with is a subset of the universal set.

To make life easier, we will call the universal set, $\mathbb{U}$. Everytime we draw a Venn Diagram, we first draw our universe like a big rectangle; we draw the sets that we are dealing with as circles inside of our rectangle.

## Complements

Once we define $\mathbb{U}$, we can define the **complement** of any of its subsets. Let's say our universal set $\mathbb{U}$ is the first line of a hockey team.

$\mathbb{U}$ = {Matthew, Mark, Luke, John, Mary, Patricia}

John and Luke are defensemen. So let D = {John, Luke}.

Patricia is the goalie, so let G = {Patricia}.

Mary, Matthew, and Mark are the forwards, so F = {Mary, Matthew, Mark}.

We then define the complement of any subset in the universe to be all elements that are NOT in the subset but ARE in the universe. If we call our subset $X$, we write this as $X^c$. The "c" is written as a **superscript** - kind of like an exponent.

Then, $D^c$ = {Matthew, Mark, Mary, Patricia}. We are going to define $\mathbb{U}^c = \phi$ (what's outside of the universe? Nothing!)

You can also find the complement of the union or intersection of two sets.

## Checkup

1. Write out the set $G^c$. Draw a Venn Diagram using only $G$ and the universe.

2. Figure out the set $(D \cup F)^c$. Draw a Venn Diagram and shade this area in.

**The Inclusion Exclusion Principle**

A survey was conducted of 28 students who buy lunch at Mathis Funn High School.

- 15 of them like pizza.
- 8 like burgers.
- 3 like both.

Organize this into a Venn Diagram. How many students like one or the other?

Notice that our answer, 20, could also be obtained by adding all the people who liked pizza and all the people who liked burgers, and then subtracting those who like both. Why can we do this?

Let's first make a definition. For any set (let's call it $X$), we say the **size** of the set is how many elements it has, and we write this as $\#X$. So if the set of people who liked pizza was called $P$, we would say $\#P = 15$. Let's call the set of people who like burgers $B$; then of course, the set of people who like both is $P \cap B$.

What kind of elements belong to $P \cup B$, in plain English? The people who like one or the other! Therefore, since we want to know the NUMBER of people who like one or the other, our question can be asked mathematically - we are looking for $\#(P \cup B)$.

When we add together the number of people who like pizza ($\#P$) to the number of people who like burgers ($\#B$), we are **double counting** the people who like both. (They got counted once in the people who like pizza, and they are counted again in the people who like burgers). So we have to subtract the number of people who like both!

$$\#P + \#B - \#(P \cap B) = \#(P \cup B)$$

This is called the **"Inclusion-Exclusion Principle"**.

# Problem Set

### Part A - Concepts and Basics

1. Is $\{1,2,3,1,5,9,10\}$ a valid set based on what we've learned? Why or why not?
2. Is $6 \in$ {the set of all even numbers}?
3. Is the set $\{a,b,f,d\}$ contained in the set $\{a,b,c,f\}$?
4. If $\mathbb{U} = \{1, 2, 3, 4, 5\}$, $B = \{5,2,3,1\}$, $A = \{5\}$
   (a) Draw a Venn Diagram to illustrate containment ($A \subseteq B$).
   (b) Find $A \cup B$. Draw a Venn Diagram illustrating this.
   (c) Find $A \cap B$. Draw a Venn Diagram illustrating this.
   (d) Find $A^c$ and $B^c$
   (e) What is $(A^c)^c$? What is $(B^c)^c$?
   (f) For any set $X$, what is $(X^c)^c$?
5. * If our universe $\mathbb{U}$ is the set of all positive whole numbers $\{1,2,3,4,5,\ldots\}$
   (a) Is $\{2,4,6,8,10\} \subseteq \{1,2,3,4,5,\ldots\}$?
   (b) Is the set of all odd positive numbers a subset of our universe?
6. What is $\{\} \cup \{\}$? Explain why in English.
7. * What is $\{\} \cap \{\}$? Explain why in English.

Let R = {Euclid, Riemann, Newton, Hilbert}, S = {Galois, Gauss, Abel, Euclid}, and T = {Hilbert, Riemann, Godel, Wiles, Chern}.

8. Let $\mathbb{U} = (R \cup S) \cup T$. What is this set?
   (a) Using this as our universe, what is $R^c$?
   (b) What is $(S \cap T)^c$? What is $S^c \cup T^c$?
9. What is the set $(R \cup S) \cap (S \cap T)$?
10. Using {R, S, T, ∩, ∪}, fill in the blanks to complete the equation:

$$(\_\_\_\_ \quad \cap \quad T) \quad \_\_\_\_ \quad (S \quad \_\_\_\_ \quad R) = \{\text{Riemann, Euclid, Hilbert}\}$$

**Part B - Applying Inclusion Exclusion Principle**

1. 40 students were surveyed about their favourite singer.

   - 24 liked Justin Bieber.
   - 12 liked Miley Cyrus.
   - 7 liked both.

   How many students like one or the other?

2. Sherlock Holmes has just called you in to solve a difficult mystery. A group of thieves are getting ready to rob the bank, but he does not know exactly how many will do the job.

   - 17 of them are known safecrackers.
   - 28 of them are known getaway drivers.
   - There are 40 thieves who are safecrackers and/or drivers.

   Holmes knows that the group that will rob the bank will be those thieves who are both safecrackers and getaway drivers, but he didn't pay attention in math class and can't figure it out. How many people will be in this group?

3. * There has been a mishap at the Hogwarts Guidance Office. A clumsy clerk has mixed up all of the Sorting Hat Data for 21 students. She was able to salvage the following data:

   - 13 of them are enrolled in Potions.
   - 12 are enrolled in Transfiguration.
   - 16 are enrolled in Charms.
   - 8 of them are enrolled in Potions and Transfiguration.
   - 7 are enrolled in Charms and Transfiguration
   - 9 are enrolled in Charms and Potions
   - 4 are enrolled in all three classes.

   Draw a Venn Diagram of the situation. You can assume your universe is the set of all Hogwarts students. Name each of your sets with a letter in your diagram.

   How many students are enrolled **only** in Potions?

4. ** You know the formula for the Inclusion-Exclusion Principle of 2 sets... can you derive a similar formula for 3 sets? **Can you prove why this is true?**