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Centre for Education in
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Grade 7/8 Math Circles
Let's Learn to Count

PROBLEM SET SOLUTIONS

1. 16!

2. $16 \times 15 \times 14 \times \cdots \times 3 = \dfrac{16 \times 15 \times 14 \times \cdots \times 3 \times 2 \times 1}{2 \times 1} = \dfrac{16!}{2!}$

3. In this case, $n = 7$, $k = 3$, $(n - k) = 4$, so ${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!}$

4. $n = 7, k = 5, (n - k) = 2$ so $\binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{7!}{5!2!}$

5. The order MATTERS!!! So we need to use our formulas for permutations. Since we cannot repeat runners, we have to use the "pick" formula. We have 5 runners to choose from - this means $n = 5$. We are using 5 of them - this means $k = 5$. So the answer is ${}_5P_5$, which is just 5!.

   If we used the fundamental counting rule, we would have gotten the same answer, since we have 5 choices for the first runner, 4 for the second (we already used one runner for first), and so on, which gives $5 \times 4 \times 3 \times 2 \times 1 = 5!$.

6. Order DOES not matter since you are just picking groups. In this case, we use the "choose" formula, for combinations. There are 5 students to choose from, so that means $n = 5$; we are choosing 2 of these 5, so $k = 2$. Therefore, the number of possible combinations is ${}_5C_2$.

7. The question states that position DOES NOT matter, so we are using combinations. He has 12 players to choose from, and must choose 5 of them. This

means the answer will be $\binom{12}{5}$.

8. When you assign positions - order MATTERS!

   Imagine 12 blanks like this: __ __ __ __ __ __ __ __ __ __ __ __, where each blank stood for a position. Then if you fill in the blanks with player names, you are assigning them to the position that the blank represented.

   How many ways can you fill in these blanks? For the first blank, you can fill it in with 12 players. For the second blank, you can fill it in with 11, because you already filled the first blank with a player. Continuing on until you use all 5, the answer is ${}_{12}P_5!$ ways to assign the players.

9. Order does matter - you are being asked to arrange them. A different order is a different arrangement. You have 5 books to choose from, and you have to arrange 3 of them. This is a permutation. Therefore, we use the "pick" formula to give us ${}_5P_3 = 60$

10. Whenever we are counting the possibilities for numbers or words, the order MATTERS. We have a 6 letter word. We have 26 letters to choose from, and we are rearranging 6 of them to form a word, but we are allowed to reuse letters. Therefore, by the fundamental counting rule, we have 26 choices for the first, 26 choices for the second, 26 choices for the third, and so on. This is just $26 \times 26 \times 26 \times 26 \times 26 \times 26 = 26^6$.

    Therefore there are 26^6 possible passwords.

11. Since you cannot repeat digits, just like the password problem, this can be solved using fundamental counting rule. For the first digit, you have 4 choices; for the second, you have 3, and so on... so the answer is $4 \times 3 \times 2 \times 1 = 4! = 24$.

12. You are now allowed to repeat digits. Using the fundamental counting rule, the first digit has 4 choices, the second 4 choices (because you can repeat digits), and so on, until the sixth digit, which also has 4 choices. This means the answer is $4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4096$.

13. This is very similar to the example involving 4 girls and 4 boys. We first find out how many possibilities there are for the first 4 symbols, and how many possibilities for the last 3, and then multiply them together.

    For the first 4 symbols, using digits from 0-9, there are 10^4 possibilities (10 choices of digit for each of the four digits, and we are allowed to repeat).

    For the last 3 symbols, it is the same idea, but instead we have 26 letters. So this is just 26^4.

    Multiplying these two together, we get $10^4 \times 26^4$.

14. Since the number must be even, the last digit only has 5 choices. The first digit only has 9 choices (0 cannot be used in the first position). The 2nd, 3rd, 4th, 5th can be any of the 10 digits. So the answer is, by fundamental counting rule, $9 \times 10^4 \times 5 = 45000$.

15. This problem could have been approached in two ways. This is a challenge problem and thus would require you to extend your thinking.

    The first way is to count all the numbers that have EXACTLY one repeated digit, and then count all the numbers that have EXACTLY two repeated digits. This is quite difficult as there are many factors to consider.

    Instead, we will take a different approach. Notice that if you look at all four digit numbers, either they have no repeating digits, or they have at least one repeating digit. So if you could figure out how many four digit numbers there were in total, then take away how many four digits numbers with no repeating digits, you would be left with all four digit numbers with at least one repeating digit.

    So, let's figure out how many four digit numbers in total first. This is just $9 \times 10 \times 10 \times 10 = 9000$, since the first digit can't be 0.

    Now, how many four digit numbers are there with no repeating digits? You have 9 choices for the first digit. The second digit cannot be the first digit, but it can be anything else, so there are 9 choices. Then you have 8 choices left for the third, and 7 choices left for the last one. So $9 \times 9 \times 9 \times 7 = 5103$ four digit numbers without repeating digits.

    Then $9000 - 5103 = 3897$ is the answer.

16. This one required you to understand the factorial representation of the choose notation.

Notice that $\binom{n}{2} = \frac{{}_nP_2}{2!} = \frac{n \times (n-1)}{2}$.

Set this equal to 55. Then $\frac{n \times (n-1)}{2} = 55$, or $n \times (n-1) = 110$. So now you need to solve for n. This requires a little bit of guesswork. What two consecutive numbers, when multiplied by each other, give 110? It's pretty clear - 11 and 10 are consecutive numbers, and 11 times 10 is 110. So $n = 11$.

17. You could have approached this problem in two different ways. You could have just used basic reasoning and realized the first person had to shake hands with 9 people; the second person shakes hands with 8; the third with 7... and so on, until the last person, who doesn't shake hands at all. The total number of handshakes is just $9+8+7+6+5+4+3+2+1 = 45$.

    Another, more clever way to approach this problem is to realize that when you count the number of unique handshakes, you are really counting the number of possible pairings between people. How many possible pairings are there? This is the same as asking - if I have 10 people, how many ways can I CHOOSE two of them? This then gives the answer ${}_{10}C_2$ - if you evaluate, this will equal 45, as before.

18. This problem is the exact same as the previous one! When you draw a diagonal, you are counting a unique handshake! Therefore, the answer is just ${}_{10}C_2$, since you have 10 corners (people) and are choosing two of them to draw a diagonal (shake hands).

19. $\binom{16}{8}$.

20. (a) To count the number of straights, first ask - "How many ways can you pick the first card?". In a straight, there are 10 possible choices for the number of the first card, and 4 possible ways you can pick the suit. So there are 40 possible ways you can take the first card. Once the first card is picked, the number of the other 4 are already determined. You just need to pick their suits. Since the suit does not matter, there are 4 choices of suit for the 2nd card, 4 choices for the 3rd, 4 for the 4th, and 4 for the 5th. Using the fundamental counting rule, the total number of straight is then $40 \times 4^4 = 10240$.

(b) We already counted the number of straight flushes in the previous answer. When you count a straight flush, you only need to pick the first card; the other 4 cards are already determined by it. So there 40 choices for the first card, and hence 40 possible straight flushes.

21. To count the full house, we need to pick the number that will make up the three of a kind, and the number that will make up the pair.

There are 13 numbers to choose for the pair, and there are (4 choose 2) ways to pick the suits for the pair. Similarly, 12 ways to pick the triple, and (4 choose 3) ways to pick the suits. Multiplying, you get

$$13 \times \binom{4}{2} \times 12 \times \binom{4}{3} = 3744$$

The two pair is slightly more tricky. One way to approach it is to select the three different numbers; two of these will be the pairs, one will be the single card. This is done in 13 choose 3 ways.

For the first pair, there are 4 choose 2 ways to select the suits. For the second pair, there are also 4 choose 2 ways to select the suits. For the lone card (kicker), there are 4 ways to choose the suits.

However, two of the three numbers need to be chosen to be the two pairs (we haven't assigned which cards will form the pair and which card won't). So you need to multiply by (4 choose 3) = 3, to account for all the possible assignments. The answer is then

$$\binom{13}{3} \times \binom{4}{2}^2 \times 4 \times 3 = 123552$$

Another way, similar to the one for the full house, is as follows: You have 13 choices of number for the first pair, 12 choices of number for the second pair, and 11 choices for the kicker. There are 4 choose 2 choices of suit for the first pair, 4 choose 2 choices of suit for the second pair, and 4 choices of suit for the kicker.

However, you need to also divide by two, because the two pairs could be double counted (you need to make order not matter). This gives

$$13 \times \binom{4}{2} \times 12 \times \binom{4}{2} \times 11 \times 4 \div 2 = 123552$$