UNIVERSITY OF
WATERLOO
Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles
Let's Learn to Count!
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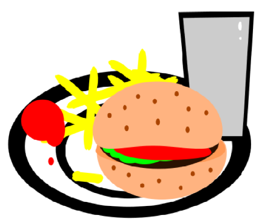
Counting is a basic building block of mathematics - but it is also one of it's most challenging, fun, and exciting aspects as well!

The fundamental purpose of counting is to **keep track of things**!

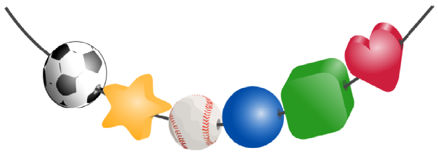
Example. You and some friends want to go the new fast food burger place in town.

The menu gives you 3 choices for burgers, 2 choices for fries, and 3 choices of pop. How many different combos can you make?



Definition. (Fundamental Counting Rule)**.** If there are a ways to pick something **and** b ways to pick something else, then in total, there are $a \times b$ total ways you can pick both!

Example. Alison wants to make a bracelet out of coloured beads. She has 6 beads, each a different colour. How many different bracelets can she make, if she uses all the beads?



Example. How many bracelets can Alison make if she has 30 different coloured beads and wants to use all of them?

Definition. (Factorial Notation)**.** A product of the form

$$n \times (n-1) \times (n-2) \cdots \times 2 \times 1$$

can be written shorthand as $n!$ - this is read as "n factorial".

In these examples, the order is important - different order, different item.

Definition. (Permutation)**.** A permutation is an arrangement of a certain number of distinct types of items, where the **order** of the items matter.

More Examples

- Numbers - the arrangement of the digits matter (repetition allowed)
- Locker "combinations" (should be permutations)
- Winners in a marathon - the order of the runners matters!

Factorial notation and exponents can help us count permutations a lot easier.

Example. How many bracelets can Alison make if she has 30 different coloured beads and wants to use 8 of them? 25 of them?

A closer look at the example...

Alison had 30 beads but only wants to use 25 of them (so she is NOT using (30 - 25) = 5 of them). Using the fundamental counting rule, we had the following expression:

$$30 \times 29 \times 28 \times \cdots \times 6$$

We can write this in a different way:

$$\underbrace{30 \times 29 \times 28 \times \cdots \times 6 \times [5 \times 4 \times 3 \times 2 \times 1]}_{30!} \div \underbrace{[5 \times 4 \times 3 \times 2 \times 1]}_{5!=(30-25)!}$$

So what we get is actually $30! \div (30-25)! = 30! \div 5! = \frac{30!}{5!}$. This is a shorter way of writing out that long expression!

What did we do? We just divided (# of objects we have)! by (# of objects we did not use)! How many objects did we NOT use? Simple - just subtract the number of objects you DID use (k) from the number of objects you didn't use (n). This is just $(n-k)$.

If we are counting permutations of n DISTINCT objects, where we use k of them **without** reusing an object, the amount is "n pick k"

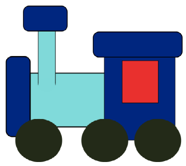
$$\text{Total Permutations} = \frac{n!}{(n-k)!} = \frac{(\text{\# of distinct objects you HAVE})!}{(\text{\# of distinct objects you AREN'T using})!} = {}_nP_k$$

(If we use ALL the objects, where $n = k$, then we just use $n!$ - we define $0! = 1$)

For permutations of n DISTINCT objects, where we use k of them and **are allowed to reuse** an object, the number is simply "$\underbrace{n \times n \times \cdots \times n}_{\text{k times}} = n^k$".

The key to remember here is - all of these formulas come from the Fundamental Counting Rule!

Example. Collector Christian has a collection of 7 toy train cars and creates a train using 4 of them. How many different ways could he have made the train?



Example. Forgetful Frank recently bought a 4 digit permutation lock. Being forgetful, he forgot the proper permutation. Being very patient, he decides to try all possible combinations, one by one. How many possibilities must he try before he cracks the lock, assuming the digits are 0-9, and you can repeat digits? (For example, 7775 is allowed).

Now you try...

1. You have 3 tops, 2 pairs of pants, and 6 pairs of socks. How many different outfits can you wear (ignoring fashion sense)?

2. How many 10 digit phone numbers are possible using digits from 0 - 9?

3. How many 10 digit phone numbers are possible using digits from 0 - 9 if the last digit must be even?

4. How many different 5 card **arrangements** can you make from a standard 52-card deck?

But what if order *doesn't* matter?

How many possible 3 digit lottery tickets are there if numbers from 1-4 are used (no repeats)? List out all permutations.

$$(1,\ 2,\ 3\quad 1,\ 3,\ 2\quad 2,\ 3,\ 1\quad 2,\ 1,\ 3\quad 3,\ 1,\ 2\quad 3,\ 2,\ 1)\ \rightarrow \{1, 2, 3\}$$

$$(1,\ 2,\ 4\quad 1,\ 4,\ 2\quad 2,\ 1,\ 4\quad 2,\ 4,\ 1\quad 4,\ 1,\ 2\quad 4,\ 2,\ 1)\ \rightarrow \{1, 2, 4\}$$

$$(1,\ 3,\ 4\quad 1,\ 4,\ 3\quad 3,\ 1,\ 4\quad 3,\ 4,\ 1\quad 4,\ 1,\ 3\quad 4,\ 3,\ 1)\ \rightarrow \{1, 3, 4\}$$

$$(2,\ 3,\ 4\quad 2,\ 4,\ 3\quad 3,\ 2,\ 4\quad 3,\ 4,\ 2\quad 4,\ 2,\ 3\quad 4,\ 3,\ 2)\ \rightarrow \{2, 3, 4\}$$

Using permutation method:

Problem... $\{1, 3, 4\}$ is the same ticket as $\{1, 4, 3\}$. We double counted.

Group all permutations which use the same digits. How many are in each group?

The number of different lottery tickets is the number of **groups** of permutations that use the same digits. This is the same as making order irrelevant!

What did we do? We divided the total number by the number of elements in each group - how many are each in group? The number of permutations of k elements - this is exactly k! **So we divide by** k!

Definition. (Combination)**.** A selection of items where order does not matter is known as a **combination.** There are ${}_nP_k$ total permutations; but each **grouping** has $k!$ elements, so we have to divide by $k!$. Then there are

$$[{}_nP_k] \div k! = \frac{{}_nP_k}{k!} = \left[\frac{n!}{(n-k)!}\right] \div k! = \frac{n!}{k!(n-k)!}$$

combinations in total. We write this as $\binom{n}{k} = {}_nC_k$, and say "n **choose** k".

So the actual number of lottery tickets is $\binom{4}{3} = 4$ - from 4 possible numbers, you are CHOOSING three of them, where order DOES NOT matter.

Example. How many possible Lotto Max lottery ticket numbers are possible (7 numbers, each number goes from 1-49, no repeats)?

Example. 22 people tried out for the basketball team. How many possible 5-man starting lineups can the coach choose, disregarding position?

Example. * Ms. Schott wants to pick 8 volunteers from her class for the Math Olympics team. She wants 4 girls and 4 boys - how many different ways can she do this if there are 13 girls and 12 boys in her class?

Checkup

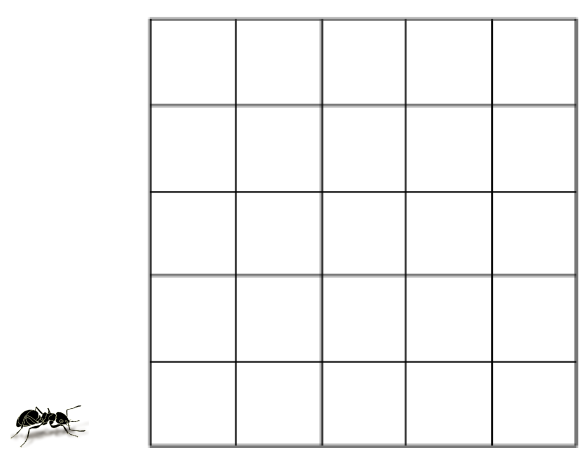
Identify whether order matters in the following. Then express the answer either as ${}_nP_k$ or ${}_nC_k$.

1. There is a bag of 12 gumballs, each of different colour. How many handfuls of 3 gumballs can you make?

2. Eve is a gardener. She has 15 different flowers that she wants to arrange in a row, but she only has space for 4 of them. How many possible arrangements can she make?

A really cool application of combinations...

An ant has been placed at the corner of a 5 x 5 maze grid. It can only go up or to the right. How many different paths can it take?



Problem Set

`NOTE`: **I do not expect you to solve problems #16, #20 and #21.** For 20 and 21, I have included the numerical answer if you want to try and solve them yourself.

Convert the following expressions into factorials or ${}_nP_k/{}_nC_k$ form:

1. $16 \times 15 \times 14 \times \cdots \times 2 \times 1$
2. * $16 \times 15 \times 14 \times \cdots \times 3$
3. ${}_7P_3$
4. $\binom{7}{5}$

For each of the following, identify whether you should be using combinations or permutations, and explain your choice. Then count the correct number of possibilities by writing out an expression in factorial or choose notation. **If the numbers aren't too big, evaluate with your calculator.**

5. 5 runners are running a marathon. In how many different ways can they finish?
6. Elections are being held for student council representative. If 5 students are running for election, how many different groups of 2 can be elected to represent the class?
7. A coach must choose five starters from a team of 12 players. How many different starting LINEUPS can he pick (where positions don't matter)?
8. * How many ways can the coach pick the 5-man lineup from the 12 players if positions do matter?
9. Your parents have bought you five new math books; unfortunately, your desk can only hold three of them. How many different ways can you arrange these books on your desk?
10. You forgot your email password... you remember that you used only letters of the alphabet (lowercase letters only), and that it was 6 letters long. How many possible passwords can you try?

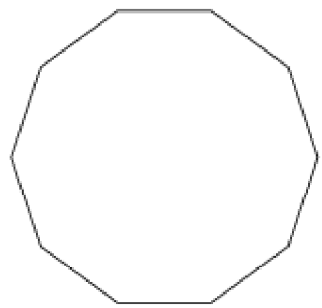
Count the number of possibilities (use fundamental counting rule!):

11. How many 4 digit numbers can you make with the digits 1, 2, 3, 4 without repeating digits?
12. How many 6 digit numbers can you make with the digits 1, 2, 3, 4 if you are allowed to repeat digits?
13. How many different 7-symbol license plates can be made if the first 4 symbols must be digits from 0-9, and the last 3 symbols must be letters of the alphabet? (Hint: Remember the Fundamental Counting Rule)
14. How many 6 digit even numbers are there with no repeating digits?
15. * How many 4 digit numbers are there with **at least one** repeated digit?
16. ** Solve for n in the following: $\binom{n}{2} = 55$

Word Problems

17. There are 10 members in the Handshaking Club. At a typical meeting, everybody shakes hands with each other, exactly once. How many handshakes occur? Can you solve this using what we learned today?

18. How many different diagonals can you draw on a regular decagon (10 sided polygon?) This is related to Question 17... can you see why?



19. In Wunwayton, all the streets are one-way pointing North, and all the avenues are one way pointing East. The entire city is made of 8 streets and 8 avenues forming a grid.

    A taxi is located at the bottom left corner of the grid, the corner of 1st Avenue and 1st Street. His passenger wants to get to the corner of 8th and 8th (top right corner). How many different routes can he take?

20. ** In poker, a "straight" is defined as 5 cards in numerical order, where the suit does not matter. (Assume Jacks are 11, Queens are 12, Kings are 13, and Aces can be used as 1 or 14 - e.g. you can have `A2345` or `10JQKA`).

    (a) How many possible straights are there? (10240)

    (b) How many possible straight flushes (a straight where all the cards are **the same suit**) are there? (40)

21. *** A **full house** is a poker hand where you have three of a kind and a pair. How many possible full house hands are there? (Answer: 3744). If you can get this, congrats! - how many possible **two pair** hands there are (two pairs, one random card, all five cards can't be same suit) (Answer: 123 552)