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Centre for Education in
Mathematics and Computing

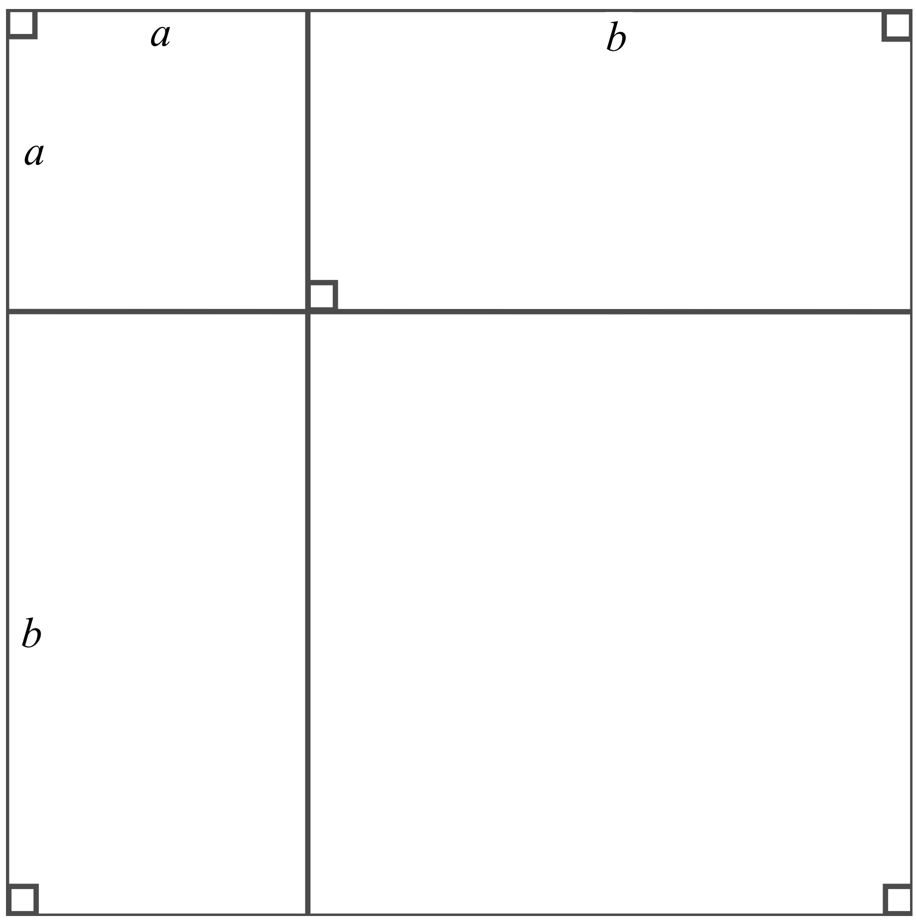
Grade 7/8 Math Circles

Geometrical Arithmetic Proofs

There were no required problems for this week. The proofs of the constructions for Multiplication, Division, Square Root, Squaring, etc... are beyond your necessary knowledge, so no proofs have been furnished. If you are interested in proofs of those, you may send me an email at ttzhou@uwaterloo.ca.

Challenge Investigation

You were given the following diagram:



The top left square is of area a^2. The top right rectangle is of area ab. The bottom right square is of area b^2. The bottom left rectangle is of area ba.
The area of the entire square is $(a+b)^2$ since the side lengths are $(a+b)$. You can also find the area of the square by summing the areas of the smaller squares: $a^2 + ba + ab + b^2 = a^2 + 2ab + b^2$. Since both should give the same area,

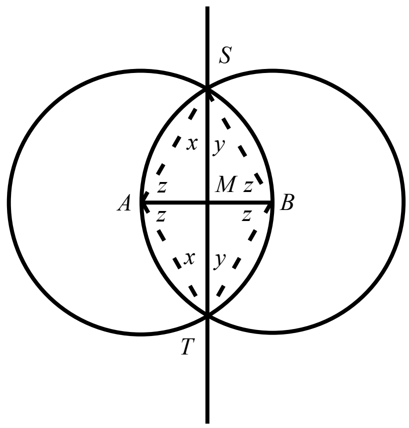
$$(a+b)^2 = a^2 + 2ab + b^2$$

as we wanted to show.

The following are proofs of the four necessary skills we did together in class. Again, **you do not need to know these**, but they are here for your own reference.

Proof of Perpendicular Bisector Construction

Consider the following configuration. This is what you would have obtained if you followed the steps properly:



Both circles have the same radius because we used the same line segment as the radius of the circle.

In the diagram, AS and AT are radii of Circle 1; BS and BT are radii of Circle 2. But both circles have the same radius length, so $AS = AT = BT = BS$.

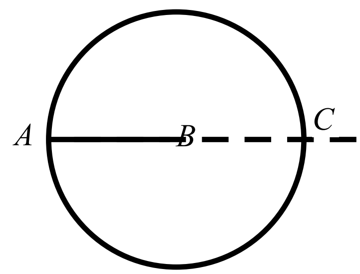
Look at $\triangle AST$. Since $AS = AT$, the triangle is isosceles. In an isosceles triangle, the base (bottom) angles are equal. The base angles are $\angle AST$ and $\angle STA$. They are equal, so $\angle AST = \angle ATS = x$. Similarly, looking at $\triangle BST$, $\angle BST = \angle BTS = y$. Therefore, $\angle BTA = x + y = \angle BSA$.

Look at $\triangle BTA$ and $\triangle BSA$. Since $BT = BS$, $TA = SA$, and $\angle BTA = \angle BSA$, then the two triangles are congruent. So ALL their angles are equal, and hence
$\angle SAB = \angle SBA = \angle TAB = \angle TBA = z$.

But now the angle between BS and BT is the same as the angle between AS and AT ($2z$). Since $BS = ST = AS = AT$, $\triangle BST$ is congruent to $\triangle AST$. So $x = y$, in fact.

Look at $\triangle SMA$ and $\triangle SMB$. They share a common side SM, and $SB = SA$, so these are corresponding sides that are equal. Also, the contained angle between the corresponding sides are equal. Hence $\triangle SMA$ and $\triangle SMB$ are congruent. That means $\angle SMA = \angle SMB$ and $AM = MB$. But $\angle SMA + \angle SMB = 180°$ (straight line). So $\angle SMA = \angle SMB = 90°$, so the line is perpendicular. Also, since $AM = MB$, it must mean the line cut AB in half.

Proof of Perpendicular Through Endpoint Construction

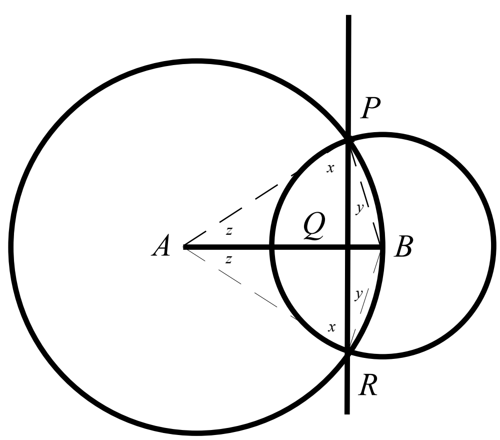


Given the segment AB, you extended it to the right. This is the dotted section of the line.

You were asked to draw a circle of radius AB with center at B, which is the circle in the picture. Where the circle intersected the line was marked as C, as in the diagram. C is on the circle, so BC is a radius. BA is also a radius, so $BA = BC$.

Then B is the midpoint of the segment AC. The rest of the construction had you simply draw the perpendicular bisector of AC. Since B is the midpoint of AC, that line will pass through B. Therefore, you make a line that is perpendicular to AC which passes through B.

Proof of Perpendicular Through Point Not on Line



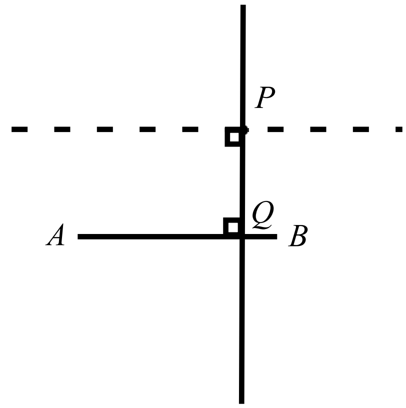
If you did not erase any of your circles, you would have the above diagram when you were finished. $PB = BR$ since they are radius of the same circle. $PA = AR$ since they are radius of the other circle. So $\triangle APR$ and $\triangle BPR$ are isosceles. Therefore, the base angles are the same, so we label $\angle BPR = \angle BRP = y$ and $\angle APR = \angle ARP = x$.

Now, look at $\triangle APB$ and $\triangle ARB$. In these two triangles, they have corresponding sides $AP = AR$ and $PB = RB$. Also, the angle contained between them are equal, since $\angle ARB = \angle APB = x + y$. Therefore, both triangles are congruent. Then $\angle PAB = \angle RAB = z$.

In $\triangle AQP$, $z + x + \angle PQA = 180°$. In $\triangle AQR$, $z + x + \angle RQA = 180°$.

So it must be that $\angle RQA = \angle PQA$. But they also form a straight line. In a straight line, $\angle RQA + \angle PQA = 180°$. This means that $\angle RQA = \angle PQA = 90°$. So the line PR is perpendicular to AB like we wanted.

Proof of Perpendicular Through Point Not on Line



This construction tied together everything you learned.

The line PQ was constructed perpendicular to the original line segment AB. This line passed through P when you constructed it.

The next steps were to construct a line through P that was perpendicular to PQ. So you would get the dotted line in the diagram. Notice that both of the angles are 90 degrees, by construction. By definition, those two lines are parallel.