Math Circles - Sequences and Series 2

Wednesday, April 3, 2013

Last week we saw many examples of sequences. One thing we didn't talk about was the concept of convergence of a sequence. Without getting too formal (you'll see that in university courses), we say that a sequence **converges** (or "the sequence is convergent") if the terms are all approaching (or remaining at) a particular finite number (note: the word "finite" means "not infinite" if you were not aware of this fact). Otherwise, we say that the sequence **diverges**. If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a number L, we might write $a_n \overset{n\to\infty}{\longrightarrow} L$.

Examples: The sequence $a_n = \frac{1}{n}$ converges to 0. The sequence $a_n = 2^n$ diverges.

We we will need this concept when discussing series, which is the next topic.

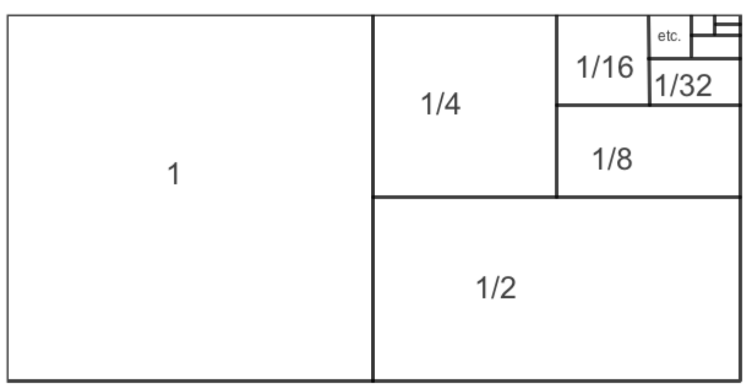
Given a sequence $\{a_n\}_{n=1}^{\infty}$, we define the related **series** to be the sum of all of the terms in the sequence. For most sequences, there are infinitely many terms, so this creates a bit of a problem. We can't exactly add up infinitely many things, at least not directly. To get us started, let's consider the sequence $\{\frac{1}{2^n}\}_{n=1}^{\infty}$. That is, the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

The sequence itself clearly converges to 0, but what about the series? Does the infinite sum

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

actually add up to something other than ∞? There are infinitely many terms being added, so it seems like they might accumulate and the sum could become huge. On the other hand, the terms being added are getting progressively smaller, so maybe the tininess of the terms balances out the fact that there are infinitely many of them, and maybe the sum could be finite after all. We will present a geometrical argument for why this sum is actually finite (and we'll find its value) below:



The picture shows a square of area 1, and then next to it is a block half of its size (area = 1/2). Above this 1/2 rectangle, we look at half of THAT (which would be 1/4) and place it above it. We continue this pattern, and we can see that adding half of the previous area will gradually fill in one entire second square of area 1. That is, if we really "add" all infinitely many of these areas, the total area will simply be 2. Cool!

A similar analogy is if I am standing 2 metres from a wall, I can start walking towards the wall by first taking a step of length 1 metre, and then taking a step of length 0.5 metres, then half of that (0.25 metres), and continue the pattern. I will get closer and closer to the wall, and in my infinitely many steps, I will have covered the entire distance of 2 metres.

In both cases, we see that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=2.$$

SIGMA SUMMATION NOTATION

Before going any further, let's introduce the so-called "sigma notation" representing sums. This is just so that we can talk about these topics with a bit more ease. The symbol

$$\sum_{n=1}^{m} a_n$$

is a compact notation meaning

$$a_1+a_2+\cdots+a_m$$

The symbol Σ is the Greek letter "sigma", which we remember has the same first sound as the word "sum". The "$n=1$" on the bottom of the Σ tell us that n is a variable which is being initiated at 1, and the m on the top says that the last number we will reach is m. It is understood that we increment the expression a_n by 1 and add each term.

The n itself is essentially a placeholder variable (not actually appearing in the final expanded version of the sum), so we could actually use any variable. That is,

$$\sum_{n=1}^{m} a_n=\sum_{i=1}^{m} a_i=\sum_{j=1}^{m} a_j;$$

we can use any letter for the "index" that we'd like. The "$n=1$"

More examples to make this make more sense:

$$\sum_{n=5}^{8}\frac{5n}{4}=\frac{5\cdot 5}{4}+\frac{5\cdot 6}{4}+\frac{5\cdot 7}{4}+\frac{5\cdot 8}{4}=\frac{5}{4}(5+6+7+8)=\frac{5}{4}(26)=\frac{65}{2}.$$

Notice in the above we factored out the 5/4, so we could also think of the sum as $\frac{5}{4}\sum_{n=5}^{8} n$.

$$\sum_{k=3}^{4}\frac{k^2}{\sqrt{k}}=\frac{3^2}{\sqrt{3}}+\frac{4^2}{\sqrt{4}}=\frac{9}{\sqrt{3}}+8$$

$$\sum_{n=7}^{7}\frac{n}{n^2+1}=\frac{7}{50}$$

$$\sum_{i=1}^{10} 4=4+4+4+4+4+4+4+4+4+4=40$$

Now that we understand this Σ notation, we can proceed with the discussion of series. Simply put, a series (being the sum of the terms in a sequence) is just

$$\sum_{n=1}^{\infty} a_n.$$

As mentioned before, an "infinite sum" isn't really easily handled as-is. We need to approach this another way. We define the nth **partial sum** to be the sum of the first n terms in a sequence, and we denote the nth partial sum by s_n. That is,

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$\vdots$$

$$s_n = a_1 + a_2 + \cdots + a_n$$

Note that we can also write s_n as

$$s_n = \sum_{k=1}^{n} a_k$$

Now look at what we've done. We have created a *sequence* of partial sums, $\{s_n\}_{n=1}^{\infty}$. And then if the *sequence* $\{s_n\}_{n=1}^{\infty}$ converges to a finite number (say L), then we say that the infinite series has a sum of L. That is, if $s_n \overset{n\to\infty}{\longrightarrow} L$, we write $\sum_{n=1}^{\infty} a_n = L$. We would also say that the *series* **converges**. If the sequence of partial sums diverges, we say that the series diverges.

So now we have a way to handle these infinite sums: We come up with some formula for the nth partial sum s_n, and we check if the sequence $\{s_n\}_{n=1}^{\infty}$ converges.

Let's go back to the example we looked at ($a_n = \frac{1}{2^n}$) and, in fact, look at it more generally and formally.

Consider the series

$$\sum_{n=0}^{\infty} r^n$$

. That is,

$$1 + r + r^2 + r^3 + \cdots$$

Does this converge? The previous example was exactly this series with $r = \frac{1}{2}$, and we saw an intuitive explanation that it does converge, so we should be able to show this more formally somehow. As well, you can probably guess that the answer as to its convergence depends on the value of r. If $r = 1/2$, the series converges. But if $r = 1$ (for example), we get

$$1 + 1 + 1 + \cdots ,$$

which clearly does NOT converge. How can we figure out what happens for all possibilities of r?

Well, consider the following:

The nth partial sum is $s_n = 1+r+r^2+\cdots+r^n$. And then r times s_n is $rs_n = r+r^2+\cdots+r^{n+1}$. If we subtract:

$$s_n - rs_n = 1 + r + r^2 + \cdots + r^n - (r + r^2 + \cdots + r^{n+1})$$

we simply get

$$1 - r^{n+1}$$

(all the other terms cancel with both positives and negatives appearing). As well, we can factor $s_n - rs_n = s_n(1-r)$ so that we have

$$s_n(1-r) = 1 - r_{n+1}$$

or (as long as $r \neq 1$)

$$s_n = \frac{1-r^{n+1}}{1-r}.$$

Now we can get an idea of the different cases for what r could be. We already discussed how if $r = 1$ the series diverges, so we don't need to consider that case. If $r > 1$ it's even more obvious that the series will diverge. You can see that from the original form for the series, or by looking at the partial sum (as $n \to \infty$, the value of r^{n+1} will grow without bound). Similarly, if $r \leq -1$ the series will diverge. We can make all of this concise by saying that the series diverges whenever $|r| \geq 1$.

The other case (when $|r| < 1$, or $-1 < r < 1$) actually gives convergence. Looking at the partial sum again, as $n \to \infty$, the r_{n+1} term will get closer and closer to zero, disappearing completely if we take n as large as can be ("infinite"). That is, the series converges for $|r| < 1$, and it converges to

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

You can now view that previous example in this light by setting $r = 1/2$ and getting

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = 2$$

We can use this formula to solve (some) other series:

Example: What is $\sum_{n=1}^{\infty} \frac{1}{10^n}$?

We know that $\sum_{n=0}^{\infty} \frac{1}{10^n} = \frac{1}{1-\frac{1}{10}} = \frac{10}{9}$, but that's the sum starting at $n = 0$. Our sum starts at $n = 1$. This means that the answer we have has an extra term. We look at this as follows:

$$\sum_{n=0}^{\infty} \frac{1}{10^n} = 1 + \sum_{n=1}^{\infty} \frac{1}{10^n}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{10^n} = \left(\sum_{n=1}^{\infty} \frac{1}{10^n}\right) - 1 = \frac{10}{9} - \frac{9}{9} = \frac{1}{9}.$$

We could similarly show that

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{2}$$

(we'll leave the details to you, as an exercise).

As we discussed when this concept was introduced, some series will converge and other diverge. We saw an example where the infinite number of terms we were adding we small enough that they balanced out and we ended up with a finite sum. Will this always happen? No - it's hard to tell without doing some more thinking about it. The next example is a series where each term is getting smaller and smaller (similar to the first example), but the series will diverge, nonetheless.

Example: Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converge? Note that we start at $n = 1$ to avoid division by zero.

Let's consider a particular *subsequence* of the sequence of partial sums. That is, consider the partial sums $s_2, s_4, s_8, \ldots$:

$s_2 = 1 + \frac{1}{2}$

$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$.

Notice the inequality comes from the fact that $3 < 4$, so $\frac{1}{3} > \frac{1}{4}$.

By similar reasoning,

$s_8 = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 1+\frac{3}{2}.$

We can extend this idea and infer that

$$s_{2^n} > 1 + \frac{n}{2}$$

In particular, these terms are greater than many multiples of 1/2, which will get bigger and bigger as $n \to \infty$. We conclude that these particular partial sums diverge, so all the partial sums (which include these ones) cannot converge, so the series diverges.