Math Circles April 3, 2013

Solutions to Exercises About Series

1. Find a simple expression for the nth partial sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

and hence find its sum. *Hint: This type of series is referred to as* telescoping.

Solution:
We note that the nth partial sum has the form

$$\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\cdots+\left(-\frac{1}{n}+\frac{1}{n}\right)-\frac{1}{n+1}$$

$$=1+0+0+\cdots+0-\frac{1}{n+1}$$

$$=1-\frac{1}{n+1}=\frac{n+1-1}{n+1}=\frac{n}{n+1},$$

and as $n\to\infty$, the value of $\frac{n}{n+1}$ gets closer and closer to 1. Thus the sum of the infinite series is 1.

2. Prove that the real number 0.999999... $= 0.\bar{9}$ (i.e. the terms continue forever, with all digits equal to 9) is equal to one. *Hint: Write the number as 0.9 + 0.09 + 0.009 + ...*

Solution:
We write the number

$$0.\bar{9}=0.9+0.09+0.009+\cdots=\sum_{k=0}^{\infty}0.9\left(\frac{1}{10}\right)^{k}=\frac{0.9}{1-\frac{1}{10}}=\frac{9/10}{9/10}=1,$$

as required.

3. Write the number $2.3\overline{17} = 2.3171717...$ as a ratio of integers.

Solution:

$$2.3\overline{17}=2.3+\frac{17}{10^3}+\frac{17}{10^5}+\frac{17}{10^7}+\cdots$$

$$=2.3+\frac{17}{10^3}\left(1+\frac{1}{100}+\frac{1}{100^2}+\frac{1}{100^3}+\cdots\right)$$

$$=2.3+\frac{17}{10^3}\cdot\frac{1}{1-\frac{1}{100}}=2.3+\frac{\left(\frac{17}{1000}\right)}{\left(\frac{99}{100}\right)}$$

$$=\frac{23}{10}+\frac{17}{990}=\frac{1147}{495}.$$

4. Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent? If it is convergent, determine the sum.

Solution:

Rewrite $2^{2n}3^{1-n} = 4^n \cdot \frac{3}{3^n} = 3\left(\frac{4}{3}\right)^n$. Since $r = \frac{4}{3}$ is greater than 1, the series diverges.

5. Is the series $\sum_{n=1}^{\infty} 2^{2n}5^{1-n}$ convergent or divergent? If it is convergent, determine the sum.

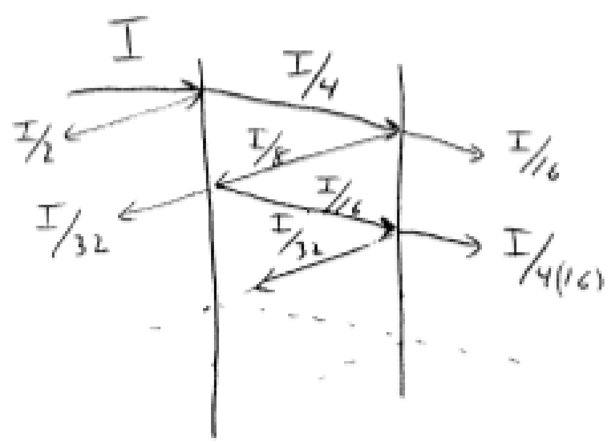
Solution:

This is the same (almost) as above and we get that the sum is

$$\sum_{n=1}^{\infty} 2^{2n}5^{1-n} = \sum_{n=1}^{\infty} 5\left(\frac{4}{5}\right)^n = 5\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n = 5 \cdot \frac{1}{1-\frac{4}{5}} = 5 \cdot 5 = 25$$

6. When light hits a certain pane of glass, the glass reflects one half of the light, absorbs one fourth, and transmits one fourth. A window is made of two panes of this glass separated by a small gap. If light of intensity I shines directly onto the window, what fraction is transmitted to the other side of the double pane?

Solution:



The amount of light transmitted through both panes is

$$\frac{I}{16} + \frac{I}{4(16)} + \frac{I}{4^2(16)} + \cdots$$

$$= \frac{I}{16}\sum_{k=0}^{\infty}(1/4)^k = \frac{I}{16}\left(\frac{1}{1-1/4}\right) = \frac{I}{16}\left(\frac{1}{3/4}\right) = \frac{4I}{3(16)} = \frac{I}{12}$$

One twelfth of the incident light is transmitted.