# Intermediate Math Circles
## Analytic Geometry I

## Problems and Solutions

1. Three points are *collinear* if they all lie on a straight line. Show that $P(-12,1)$, $Q(-4,-3)$ and $R(6,-8)$ are collinear.

   (a) Use a slope argument to show collinearity.

   Using $m = \frac{y_2-y_1}{x_2-x_1}$, $m(PQ) = \frac{-3-1}{-4+12} = \frac{-4}{8} = \frac{-1}{2}$ and $m(QR) = \frac{-8+3}{6+4} = \frac{-5}{10} = \frac{-1}{2}$. Since $m(PQ) = m(QR)$ and the two line segments share a common point, $P$, $Q$, and $R$ are collinear.

   (b) Use a distance argument to show collinearity.

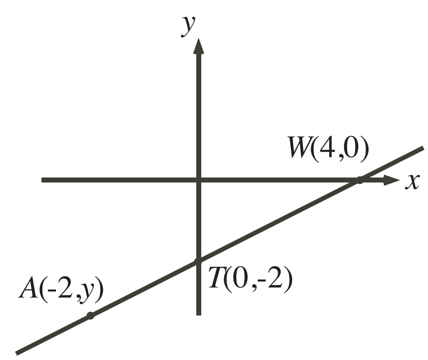
   Using $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$, we can calculate the length of the three line segments $PQ$, $QR$ and $PR$.

   $PQ = \sqrt{(-4+12)^2 + (-3-1)^2} = \sqrt{8^2 + (-4)^2} = \sqrt{64+16} = \sqrt{80} = 4\sqrt{5}$
   $QR = \sqrt{(6+4)^2 + (-8+3)^2} = \sqrt{10^2 + (-5)^2} = \sqrt{100+25} = \sqrt{125} = 5\sqrt{5}$
   $PR = \sqrt{(6+12)^2 + (-8-1)^2} = \sqrt{18^2 + (-9)^2} = \sqrt{324+81} = \sqrt{405} = 9\sqrt{5}$
   $PQ + QR = 4\sqrt{5} + 5\sqrt{5} = 9\sqrt{5} = PR$. The triangle inequality law says that $PQ + QR \geq PR$ and that the equality holds only when the three points are collinear. Since $PQ + QR = PR$, then the three points, $P$, $Q$, $R$ are collinear.

2. The point $A(-2, y)$ is on a line that passes through the points $T(0,-2)$ and $W(4,0)$. Determine the value of $y$.

   Solution 1: Using the fact that the slope of every segment on a line is the same,

$$\begin{aligned} m(TA) &= m(TW) \\ \frac{y+2}{-2-0} &= \frac{0+2}{4-0} \\ \frac{y+2}{-2} &= \frac{2}{4} \\ 4y+8 &= -4 \\ 4y &= -12 \\ \therefore y &= -3 \end{aligned}$$




   Solution 2:

   Determine the equation of the line using points $T$ and $W$. Then substitute $x = -2$ to determine the value of $y$.

   $T$ is on the $y$-axis so the $y$-intercept is $b = -2$. The slope of the line is $m = \frac{0+2}{4-0} = \frac{1}{2}$. The equation of the line is $y = \frac{1}{2}x - 2$.

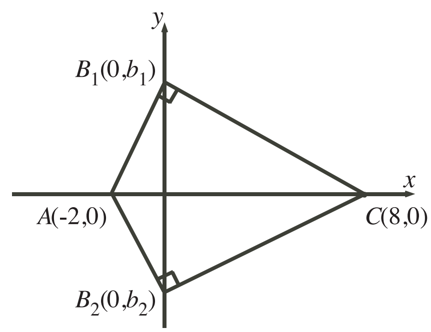
   Substituting $x = -2$ into $y = \frac{1}{2}x - 2$, $y = \frac{1}{2}(-2) - 2 = -1 - 2 = -3$. Therefore, $y = -3$.

3. $\triangle ABC$ has vertex $A$ on the $x$-axis at $-2$ and vertex $C$ on the $x$-axis at 8. The third vertex $B$ is on the $y$-axis at $b$ such that $\angle ABC = 90°$. Determine all possible values of $b$.

   Solution 1:

   By drawing a diagram, we see that it is reasonable to expect two solutions. Let $B$ have coordinates $(0, b)$. Since $AB \perp BC$,

$$\begin{aligned} m(AB) \times m(BC) &= -1 \\ \frac{b-0}{0+2} \times \frac{b-0}{0-8} &= -1 \\ \frac{b}{2} \times \frac{b}{-8} &= -1 \\ b^2 &= 16 \\ b &= \pm 4 \end{aligned}$$




   Solution 2:

   Let $B$ have coordinates $(0, b)$. Since $\triangle ABC$ is right angled,

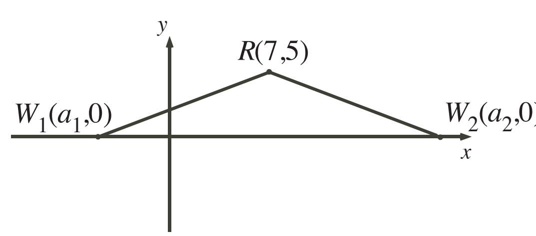
$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ (8+2)^2 + (0-0)^2 &= (0+2)^2 + (b-0)^2 + (0-8)^2 + (b-0)^2 \\ 100 + 0 &= 4 + b^2 + 64 + b^2 \\ 100 &= 2b^2 + 68 \\ 32 &= 2b^2 \\ 16 &= b^2 \\ \pm 4 &= b \end{aligned}$$

4. A point $W$ is located on the $x$-axis so that it is 13 units from the point $R(7, 5)$. Find the coordinates of point $W$.

   Solution:

   As noted in the last problem, the diagram hints that there are two possible locations for $W$. Let the coordinates of $W$ be $(a, 0)$. Using the distance formula,

$$\begin{aligned} RW^2 &= 13^2 \\ (a-7)^2 + (5-0)^2 &= 169 \\ a^2 - 14a + 49 + 25 &= 169 \\ a^2 - 14a - 95 &= 0 \\ (a-19)(a+5) &= 0 \\ a = 19 \quad a = -5 \end{aligned}$$




   $\therefore W$ is at $(-5, 0)$ or $(19, 0)$.

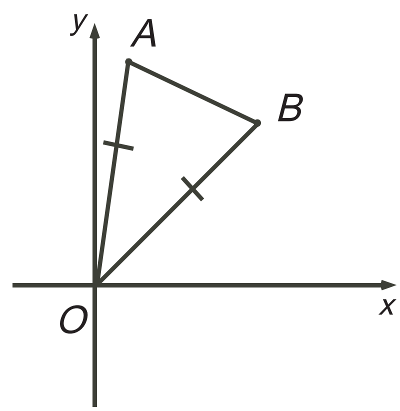
5. The points $A$ and $B$ are located in the first quadrant, equidistant from the origin, $O$. If the slope of $OA$ is 7 and the slope of $OB$ is 1, determine the slope of $AB$.

Solution:

Since the slope of $OB$ is 1, we know that $B$ is of the form $(b, b)$. We can check using the slope formula: $m(OB) = \frac{b-0}{b-0} = 1$. Also, $b > 0$ since $B$ is in the first quadrant.

Similarly, $A$ is of the form $(a, 7a)$ since $m(OA) = \frac{7a-0}{a-0} = 7$. Again, $a > 0$ since $A$ is in the first quadrant.

We know that $OA = OB$ so




$$\begin{aligned} OA^2 &= OB^2 \\ (a-0)^2 + (7a-0)^2 &= (b-0)^2 + (b-0)^2 \\ a^2 + 49a^2 &= b^2 + b^2 \\ 50a^2 &= 2b^2 \\ 25a^2 &= b^2 \\ \pm 5a &= b \end{aligned}$$

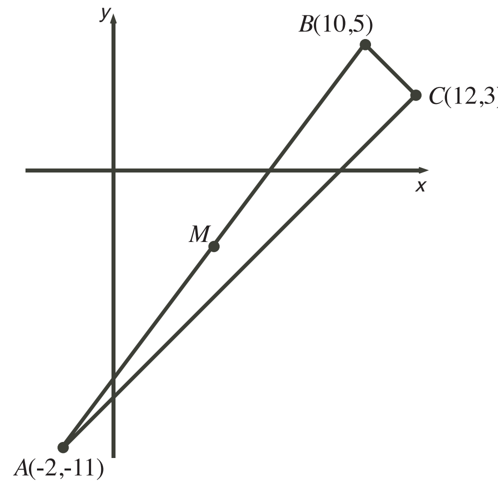
Since both $a$ and $b$ are positive, $b = 5a$ is the only solution. It then follows that $B(b, b)$ can be written $B(5a, 5a)$.

The slope of $AB$ can now be calculated using $A(a, 7a)$ and $B(5a, 5a)$.

$$\begin{aligned} m(AB) &= \frac{5a - 7a}{5a - a} \\ &= \frac{-2a}{4a} \\ &= -\frac{1}{2}, \quad \text{since } a > 0 \end{aligned}$$

Therefore, the slope of $AB$ is $-\frac{1}{2}$.

6. The vertices of $\triangle ABC$ are $A(-2,-11)$, $B(10,5)$ and $C(12,3)$.




(a) Determine the midpoint $M$ of line segment $AB$.

Using the midpoint formula, $M$ has coordinates $\left(\dfrac{-2+10}{2}, \dfrac{-11+5}{2}\right) = (4,-3)$.

(b) Show that $AM = MB = MC$. This will prove that $M$ is the centre of a circle containing points $A, B$ and $C$ on the circumference.

Using the distance formula,

$$AM^2 = (-2-4)^2+(-11+3)^2 = (-6)^2+(-8)^2 = 36+64 = 100, \ \therefore AM = 10.$$

$$BM^2 = (10-4)^2+(5+3)^2 = 6^2+8^2 = 36+64 = 100, \ \therefore BM = 10.$$

$$CM^2 = (12-4)^2+(3+3)^2 = 8^2+6^2 = 64+36 = 100, \ \therefore CM = 10.$$

$$\therefore AM = BM = CM.$$

(c) Show that $\angle ACB = 90°$.

There are several ways to show this. Two possibilities will be presented here.

$$\begin{aligned} m(BC)\times m(AC) &= \frac{3-5}{12-10}\times\frac{3+11}{12+2} \\ &= \frac{-2}{2}\times\frac{14}{14} \\ &= -1\times 1 \\ &= -1 \end{aligned}$$

Since the product of the slopes is $-1$, $BC \perp AC$ and $\angle ACB = 90°$ follows.

The second approach is to show that $BC^2 + AC^2 = AB^2$.

$$BC^2 = (12-10)^2 + (3-5)^2 = (2)^2 + (-2)^2 = 4+4 = 8$$

$$AC^2 = (12+2)^2 + (3+11)^2 = 14^2 + 14^2 = 196+196 = 392$$

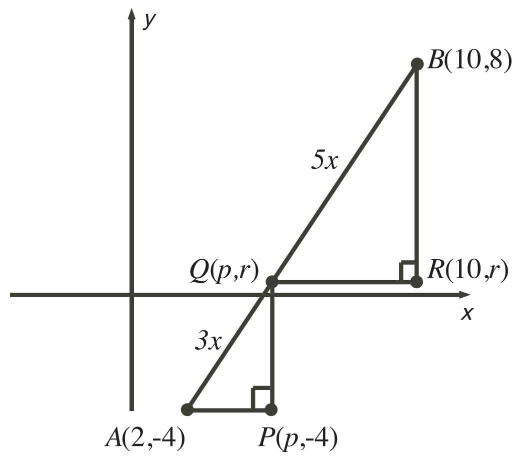
$$AB^2 = (10+2)^2 + (5+11)^2 = 12^2 + 16^2 = 144+256 = 400$$

$$BC^2 + AC^2 = 8+392 = 400 = AB^2$$

Since the converse of the Pythagorean Theorem is satisfied, $\angle ACB = 90°$, as required.

7. The line segment $AB$, where $A$ is $(2, -4)$ and $B$ is $(10, 8)$, is divided at $Q$ in the ratio $3 : 5$. Find the coordinates of $Q$.

Let the coordinates of $Q$ be $(p, r)$. Construct two similar right triangles, $\triangle APQ$ and $\triangle QRB$, as shown in the following diagram. The coordinates of $P$ would be $(p, -4)$ and the coordinates of $R$ would be $(10, r)$.




Since $AP$ and $QR$ are horizontal line segments, their lengths are $p - 2$ and $10 - p$, respectively. Similarly, since $PQ$ and $RB$ are vertical line segments, their lengths are $r + 4$ and $8 - r$, respectively.

Since $\triangle APQ \sim \triangle QRB$, the corresponding sides are in the ratio $3 : 5$. Then,

$$\begin{aligned} \frac{AP}{QR} &= \frac{3}{5} & \text{and} \qquad \frac{PQ}{RB} &= \frac{3}{5} \\ \frac{p-2}{10-p} &= \frac{3}{5} & \text{and} \qquad \frac{r+4}{8-r} &= \frac{3}{5} \\ 5p - 10 &= 30 - 3p & \text{and} \qquad 5r + 20 &= 24 - 3r \\ 8p &= 40 & \text{and} \qquad 8r &= 4 \\ p &= 5 & \text{and} \qquad r &= \frac{1}{2} \end{aligned}$$

The coordinates of $Q$ are $\left(5, \frac{1}{2}\right)$.