Intermediate Math Circles
Number Theory II

Problems and Solutions

1. The difference between the gcd and lcm of the numbers $10, 15, 35$ is
   (a) 60 (b) 205 (c) 25 (d) 1044 (e) 5245

   Method #1
   To find the $\gcd(10, 15, 35)$.
   Divisors of $10 : 1, 2, 5, 10$
   Divisors of $15 : 1, 3, 5, 5, 15$
   Divisors of $35 : 1, 5, 7, 35$
   Thus, $\gcd(10, 15, 35) = 5$.

   To find the $\text{lcm}(10, 15, 35)$.
   Multiples of 10 :
   $10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210 \cdots$
   Multiples of
   $15 : 15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180, 195, 210, \cdots$
   Multiples of $35 : 35, 70, 105, 140, 175, 210, \cdots$
   Thus, $\text{lcm}(10, 15, 35) = 210$.
   The difference of the gcd and lcm is $210 - 5 = 205$
   Answer is (b)

   Method #2
   The Prime Factorization of 10 is 2×5.
   The Prime Factorization of 15 is 3×5.
   The Prime Factorization of 35 is 5×7.
   The $\gcd(10, 15, 35) = 5$ and the $\text{lcm}(10, 15, 35) = 2 \times 3 \times 5 \times 7 = 210$.
   The difference of the gcd and lcm is $210 - 5 = 205$
   Answer is (b)

2. Two speed skaters begin practice at the same start line. The first speed skater completes one lap of the oval every 45 seconds. The second speed skater completes one lap of the oval every 63 seconds. After the start, how much time will elapse until the speed skaters are at the start line together again?

   (a) $47\frac{1}{4}$ min (b) $9\frac{9}{20}$ min (c) $5\frac{1}{4}$ min (d) $2\frac{1}{10}$ min (e) $15\frac{3}{4}$ min

   The speed skaters are at the start line together again at the $\text{lcm}(45, 63)$.

The Prime Factorization of 45 is $3^2 \times 5$.
The Prime Factorization of 63 is $3^2 \times 7$.
The $\text{lcm}(45, 63) = 3^2 \times 5 \times 7 = 315$ seconds or $5\frac{1}{4}$ minutes.

3. When a positive integer is divided by 7, the quotient is 4 and the remainder is 6. Determine the value of this number.

   Using the division algorithm, we known that a number n can be calculated using the formula $n = dq + r = 7(4) + 6 = 34$.

4. An uptown bus leaves the terminal every 70 minutes and a downtown bus leaves the same terminal every 42 minutes. If the uptown and downtown busses both leave the terminal at 9:00am, when will they be at the terminal together again?

   The busses will both be at the terminal at the $\text{lcm}(70, 42)$.
   The Prime Factorization of 70 is $2 \times 5 \times 7$.
   The Prime Factorization of 42 is $2 \times 3 \times 7$.
   The $\text{lcm}(45, 63) = 2 \times 3 \times 5 \times 7 = 210$ minutes or $3\frac{1}{2}$ hours later.
   Therefore, the uptown and downtown busses will be at the terminal at the same time at 12:30pm.

5. Erin's age when divided by $2, 3, 4, 5$ or 6 gives a remainder of 1. Determine the youngest possible age that Erin could be, if she is older than 1.

   If n is a positive integer that is divisible by $2, 3, 4, 5$ or 6, then the value of $n + 1$ will leave a remainder of 1 when divided by $2, 3, 4, 5$ or 6. The $\text{lcm}(2, 3, 4, 5, 6) = 2^2 \times 3 \times 5 = 60$. Therefore, 60 is the smallest value divisible by $2, 3, 4, 5$ or 6. Thus, 61 is the smallest positive integer greater than 1 that will give a remainder of 1 when divided by $2, 3, 4, 5$ or 6.

6. When y is divided by $2x - 4$ the quotient is $3x$ and the remainder is 9. Determine the simplified expression of y.

   Using the division algorithm, we know that
   $y = (2x - 4)(3x) + 9 = 6x^2 - 12x + 9$.

7. Determine the smallest positive integer that is divisible by both 24 and 30.

   The smallest positive integer that is divisible by both 24 and 30 is the $\text{lcm}(24, 30)$. The Prime Factorization of 24 is $2^3 \times 3$.
   The Prime Factorization of 30 is $2 \times 3 \times 5$.
   The $\text{lcm}(24, 30) = 2^3 \times 3 \times 5 = 120$.

8. Determine the smallest value of k that makes the product $48k$ divisible by 36.

   The Prime Factorization of 48 is $2^4 \times 3$.
   The Prime Factorization of 36 is $2^2 \times 3^2$.
   The $\text{lcm}(48, 36) = 2^4 \times 3^2 = 144$.
   Therefore, $k = \frac{144}{48} = 3$.

9. The student's council of your school will be selling hot dogs at the Winter Fair. Hot dogs are sold in packages of 12, but hot dog buns are sold in packages of 8. If the student's council predicts that they will need at least 150 hot dogs for the Fair, how many packages of hot dogs and buns should they buy, if every hot dog must be paired with a bun?

   The $\text{lcm}(12, 8) = 24$.
   Any multiple of 24 will match one hot dog with one bun.
   Since $\frac{150}{24} = 6.25$, the smallest multiple of 24 that is bigger than 150 is $24 \times 7 = 168$. Therefore, the number of packages of hot dogs$= 2 \times 7 = 14$ and the number of packages of buns$= 3 \times 7 = 21$.

10. A florist has 648 roses, 288 orchids and 432 tulips to create identical bouquets. What is the largest number of identical bouquets that can be created without having any flowers left over? How many flowers of each type will be in one bouquet?

    The largest number of identical bouquets is equal to the $\text{gcd}(648, 288, 432)$. The Prime Factorization of 648 is $2^3 \times 3^4$.
    The Prime Factorization of 288 is $2^5 \times 3^2$.
    The Prime Factorization of 432 is $2^4 \times 3^3$.
    The $\text{gcd}(648, 288, 432) = 2^3 \times 3^2 = 72$
    Therefore, the florist can make at most 72 identical bouquets.

    Each bouquet will have $\frac{648}{72} = 9$ roses, $\frac{288}{72} = 4$ orchids and $\frac{432}{72} = 6$ tulips.

11. Determine the $\text{gcd}(a, b)$ and lcm(a, b) for the pairs,
    (a) $(10!, 6^8)$ (b) $(130\,339, 9061)$

    (a) $(10!, 6^8)$
    The Prime Factorization of $10!$ is $2^8 \times 3^4 \times 5^2 \times 7$.
    The Prime Factorization of 6^8 is $2^8 \times 3^8$.
    The $\text{gcd}(10!, 6^8) = 2^8 \times 3^4 = 20\,736$ and the $\text{lcm}(10!, 6^8) = 2^8 \times 3^8 \times 5^2 \times 7 = 293\,932\,800$

(b) $(130\,339, 9061)$

$$\begin{array}{rl|l} & & \gcd(130\,399, 9061) \\ 103\,399 & = 14(9061) + 3485, & = \gcd(9061, 3485) \\ 9061 & = 2(3485) + 2091, & = \gcd(3485, 2091) \\ 3485 & = 1(2091) + 1394, & = \gcd(2091, 1394) \\ 2091 & = 1(1394) + 697, & = \gcd(1394, 697) \end{array}$$

Since $2(697) = 1394$, $\gcd(1394, 697) = 697$
Therefore, $\gcd(130339, 9061) = 697$.

$$\begin{aligned} \operatorname{lcm}(130\,339, 9061) &= \frac{(130\,339)(9061)}{\gcd(130\,339, 9061)} \\ &= \frac{(130\,339)(9061)}{697} \\ &= 1\,694\,407 \end{aligned}$$

12. Determine the largest value of k for $360x - 540y = k(ax - by)$ such that a, b, and k are positive integers.

    The largest value of $k = \gcd(360, 540)$
    The Prime Factorization of 360 is $2^3 \times 3^2 \times 5$.
    The Prime Factorization of 540 is $2^2 \times 3^3 \times 5$.
    The $\gcd(360, 540) = 2^2 \times 3^2 \times 5 = 180$.
    Therefore, the largest value of $k = 180$ since $360x - 540y = 180(2x - 3y)$.

13. Determine the exact value of $-\dfrac{12}{2431} + \dfrac{9}{1309}$.

    To add fractions, their denominators must be common.
    The common denominator $= \operatorname{lcm}(2431, 1309)$
    The Prime Factorization of 2431 is $11 \times 13 \times 17$.
    The Prime Factorization of 1309 is $7 \times 11 \times 17$.
    The $\operatorname{lcm}(2431, 1309) = 7 \times 11 \times 13 \times 17$

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\begin{aligned}
-\frac{12}{2431}+\frac{9}{1309} &= \frac{-12}{11\times 13\times 17}+\frac{9}{7\times 11\times 17} \\
&= \frac{7}{7}\left(\frac{-12}{11\times 13\times 17}\right)+\frac{13}{13}\left(\frac{9}{7\times 11\times 17}\right) \\
&= \frac{7(-12)}{7\times 11\times 13\times 17}+\frac{13(9)}{7\times 11\times 13\times 17} \\
&= \frac{7(-12)+13(9)}{7\times 11\times 13\times 17} \\
&= \frac{33}{7\times 11\times 13\times 17} \\
&= \frac{3\times 11}{7\times 11\times 13\times 17} \\
&= \frac{11}{11}\left(\frac{3}{7\times 13\times 17}\right) \\
&= \frac{3}{7\times 13\times 17} \\
&= \frac{3}{1547}
\end{aligned}
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