Math Circles - Sequences and Series 1

Wednesday, March 27, 2013

Suppose you obtained a month-long job (let's say this month has 31 days), and your boss gave you two choices for how you would get paid:

Option 1: You get $\$1,000,000$ for the whole month.
Option 2: You get 1 cent on the first day, 2 cents on the second day, 4 cents on the next day, 8 cents the day after that, etc. (that is, starting with 1 cent on the first day, every day you get twice as much as the day before).

Which would you choose?

Well, Option 1 doesn't require much thinking to understand. You'll get one million dollars for the job, which sounds like a lot of money (because it is!).

What about Option 2? One cent? Two cents? This doesn't sound like much. But it's not quite so simple.

Let's look at how many cents you get on each day, but introduce some notation first. Define a_n to mean the number of cents you get on the nth day. That is, $a_1 = 1$, and then $a_2 = 2$, $a_3 = 4$, etc. As mentioned above, the amount you get on one day is twice as much as the day before, so we quickly see that $a_n = 2a_{n-1}$ (for $n > 1$).

Thinking about it this way, we look at the pattern:

$a_1 = 1$

$a_2 = 2a_1 = 2$

$a_3 = 2a_2 = 2^2$

$a_4 = 2a_3 = 2 \cdot 2^2 = 2^3$

$a_5 = 2a_4 = 2 \cdot 2^3 = 2^4$

We quickly see that $a_n = 2^{n-1}$.

So let's look at how much we get on the 31st day (getting out a calculator): $a_{31} = 2^{30} = 1073741824$, which, converted into dollars (by dividing by 100) is $\$10,737,418.24$. So with Option 2, we're getting over 10 million dollars on the last day alone, not to mention all the days leading up to it. In fact, I calculated each day's pay and added them up and the total is $\$21,474,836.47$. Much more than the measly 1 million from Option 1!

The above example has some of the key elements to the subjects of sequences and series in math. Simply put, a sequence is a list of numbers, and a series is the sum of those numbers in the sequence. Let's make this a little bit more formal:

A **sequence** is an ordered list of numbers. We usually denote the first number in the sequence by a_1, the second by a_2, and in general, the nth number in the sequence is denoted by a_n, where n is a positive integer. We usually call the nth number in a sequence the nth **term** of the sequence. Usually, sequences go on forever. We sometimes abbreviate our notation and write $\{a_n\}_{n=1}^{\infty}$ to represent the whole sequence with a relatively compact symbol.

Example 1: For the sequence

$$5, 7, 4, 2, 7, 8, 3, 4, 8, 9, 3, 23, 7, 9, 9, 32, 5, 8, 9, 9, \ldots$$

the 11th term is 3, so we could write $a_{11} = 3$.

The above example isn't too interesting, because these numbers don't seem to have any pattern to them (maybe they do, but it's hard to tell). We will focus on more easily-predictable sequences.

Example 2: Consider the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 2n$. Let's write down the sequence:

$$2, 4, 6, 8, 10, 12, 14, 16, \ldots$$

Notice we have to put those dots at the end, because it would *literally* take forever to write all of the terms in the sequence.

Example 3: Let's write down the sequence $\{n^2\}_{n=1}^{\infty}$.

$$1, 4, 9, 16, 25, 36, 49, \ldots$$

Notice we changed the way we used the notation a little bit, just to show you the different ways of expressing the same thing.

Example 4: Let's write out the sequence $\{2^{n-1}\}_{n=1}^{\infty}$ from the introductory example:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, \ldots$$

Let's look at this from another perspective, though. Suppose we're given a sequence with the terms written out, and we want to know the general formula for the nth term. For example, if we are given the first three numbers in a sequence:

$$1, 4, 9,$$

we might guess that this is the sequence $\{n^2\}_{n=1}^{\infty}$. But sometimes it is ambiguous.

Example 5: What if we're given the first two numbers:

$$2, 4,$$

What is the next number? Is it 6? Are we just listing the even positive integers? Or is it 8? Are we doubling each number to get the next? Could it be 16? Are we squaring the previous term to get the next one? The answer is: It is unclear. We don't really have enough information to reasonably guess what the next term will be. Any one of these answers makes sense in their own way, but we can't really be sure. That said, if more information is given, it is often possible to guess what the pattern is. If three terms were given

$$2, 4, 16$$

it would be reasonable to guess that the next term is $16^2 = 256$.

Notice that the above sequence is defined in a very different way from the previous examples. Take the sequence in Example 2. If we were asked to find the 123rd term, it wouldn't be very difficult. The 123rd term is $a_{123} = 2 \cdot (123) = 246$. It's quite simple to find any term when there is a formula for a_n depending on n only. But in Example 5, if we were asked to find the 123rd term, it would take a longer time, since all we know is that a_{123} is a_{122}^2. But then, what is a_{122}? Well, $a_{122} = a_{121}^2$, and so on. We would need to spend a lot of time figuring out every term in the sequence for $n = 1$, $n = 2$, all the way up to $n = 122$ before we can figure out what a_{123} is. Is there a better way? Well, let's examine this a little more closely. To understand this a little better, let's look at the first few terms:

$a_1 = 2$

$a_2 = a_1^2 = 2^2$

$a_3 = a_2^2 = (2^2)^2 = 2^{(2^2)}$

$a_4 = a_3^2 = (2^{(2^2)})^2 = 2^{(2^2 \cdot 2)} = 2^{(2^3)}$

$a_5 = a_4^2 = (2^{(2^3)})^2 = 2^{(2^3 \cdot 2)} = 2^{(2^4)}$

Now it's becoming clear: $a_n = 2^{(2^{n-1})}$, so if we wanted the 123rd term in the sequence, we would just plug in $n = 123$ and get $a_{123} = 2^{(2^{123})}$. Now, it just so happens that this number is too big for our calculators, but this is beside the point. We *do* have a way of writing the nth term in general, in terms of n, which saves us the effort of having to figure out term after term after term just to get the next one.

The way that Example 5 was defined is called a **recursive** definition for a sequence. It defines a way to get a term based on a previous term, and it will also define what the first term is (because we need some place to start). Notice that the introductory example with the doubling amount of money was defined in this same recursive way. We were given that $a_1 = 1$, and that $a_n = 2a_{n-1}$ for all $n > 1$. We also went further and found a different way or representing the sequence, in the nicer way where we have a_n depending on n, but not depending on any previous terms in the sequence. This was when we figured out that $a_n = 2^{n-1}$. This is often desired with recursive sequences; to "solve the recurrence relation."

Example 6: Given the sequence $a_n = a_{n-1} + n - 1$, with $a_1 = 0$, let's solve the recurrence relation (that is, let's find a formula for a_n depending on n only). One trick for doing this is as follows: Write down the *sum* of the first n terms:

$$a_1 + a_2 + \cdots + a_{n-1} + a_n = 0 + (a_1 + 1) + (a_2 + 2) + \cdots + (a_{n-2} + n - 2) + (a_{n-1} + n - 1)$$

Now rearrange the right-hand side by putting all the a's together. Then

$$a_1 + a_2 + \cdots + a_{n-1} + a_n = a_1 + a_2 + \cdots + a_{n-2} + a_{n-1} + 1 + 2 + \cdots (n-2) + (n-1)$$

Next we notice that $a_2 + a_3 + \cdots + a_{n-1}$ appears on both sides, so if we subtract these from both sides, they'll disappear, and we're left with a_n on the left-hand side. That is,

$$a_n = 1 + 2 + \cdots + (n-2) + (n-1)$$

This is better, but still not so great, since we'd still need to add up all of these numbers just to get the nth term. Let's keep working on this and make it even better. Can we get a nice formula for a_n that doesn't require us to add many things (and more and more as n gets larger)?

Well, instead of looking at a_n, let's look at $2a_n$ (this is a trick which will make things significantly easier).

$$2a_n = a_n + a_n = (1 + 2 + \cdots + (n-2) + (n-1)) + (1 + 2 + \cdots + (n-2) + (n-1))$$

But, just for fun (or maybe there's a reason...) let's group terms in this sum two-by-two, from the outside-in. That is,

$$2a_n = (1 + (n-1)) + (2 + (n-2)) + \cdots + ((n-2) + 2) + ((n-1) + 1)$$

but then this is just

$$2a_n = n + n + \cdots + n + n$$

How many n's are there in this sum? There are $n-1$ of them, so the right-hand side is n *times* $n-1$:

$$2a_n = n(n-1)$$

Finally, we divide both sides by 2 to get:

$$a_n = \frac{n(n-1)}{2}$$

Phew! That may have been a lot of work, but NOW things are greatly simplified. If we wanted the 815th term of the sequence, we just plug in $n = 815$ and get

$$a_{815} = \frac{815 \cdot 814}{2} = 331705$$

I'm sure you'll agree that that was a lot less work than actually figuring out what the first 814 terms were and then adding 814 to the 814th term (that's what we would have had to do if we were using the recursive definition only: $a_{815} = a_{814} + 814$).