Math Circles March 27, 2013

Solutions for Exercises About Sequences

1. What is the next term in the below sequence?

$$3, 6, 10, 15,$$

Solution:

From the first term to the next term, we add 3. From the second term to the next term, we add 4. From the third to the fourth we add 5. The pattern is becoming clear. To get the next term, we should add 6 to the 4th term to get 21.

2. Consider the first few terms of a sequence:

$$3, 6, 9, 12,$$

(a) What is the next term?
(b) What is the 10th term?
(c) Write a general formula for the nth term, depending on n.

Solution:

(a) 15 (add 3 to each term to get the next).

(b) $a_1 = 3$, $a_2 = 3 + 3$, $a_3 = 3 + 3 + 3$, so we can see the pattern that $a_1 0 = 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30$.

(c) $a_n = 3n$.

3. What is the sum of $100 + 99 + 98 + \cdots + 2 + 1$?

Solution:

Although some people have a formula memorized (that $1+2+\cdots+n = \frac{n(n+1)}{2}$, we will show how to arrive at the answer as follows:

Let $S = 100 + 99 + \cdots + 2 + 1$. Then

$2S = S + S$
$= 100 + 99 + \cdots + 2 + 1$
$+1 + 2 + \cdots + 99 + 100$

$= 101 + 101 + \cdots 101 + 101$ (and there are 100 terms here)
$= 100(101)$

So $S = \frac{1}{100(101)} = 50(101) = 5050$.

4. What is the sum of $100 - 99 + 98 - 97 + 96 \cdots + 2 - 1$?

Solution:

We look at the terms 2 at a time:

$100 - 99 + 98 - 97 + 96 \cdots + 2 - 1 = (100 - 99) + (98 - 97) + \cdots (2 - 1) = 1 + 1 + \cdots 1 = 50$ (we see that there are 50 of these 1's in the sum).

5. The sum of the first n terms of a sequence is $n(n + 1)(n + 2)$.
(a) Write out the first 5 terms of the sequence.
(b) What is the 18229th term of the sequence?
(c) Write a general formula for the nth term of the sequence, a_n.

Solution:

(a) The sum of the first 1 term in the sequence is $1(1 + 1)(1 + 2) = 6$, so $a_1 = 6$.

The sum of the first 2 terms in the sequence is $2(2 + 1)(2 + 2) = 24$, so $a_1 + a_2 = 24$, but we know $a_1 = 6$, so $a_2 = 24 - 6 = 18$.

Similarly, $a_1 + a_2 + a_3 = 3(3 + 1)(3 + 2) = 60$, so $a_3 = 60 - a_1 - a_2 = 60 - 6 - 18 = 36$.

Similarly, $a_1 + a_2 + a_3 + a_4 = 4(4 + 1)(4 + 2) = 120$ so $a_4 = 120 - 60 = 60$.

And finally, $a_1 + a_2 + a_3 + a_4 + a_5 = 5(5 + 1)(5 + 2) = 210$, so $a_5 = 210 - 120 = 90$.

Thus, the first 5 terms are:

$$6, 18, 36, 60, 90.$$

(b) It actually makes more sense to do (c) before doing (b) so that we can use a more general formula. So let's do (c) first...

(c) We notice that the first n terms add up to $n(n + 1)(n + 2)$, but this also means that the sum of the first $n - 1$ terms should add up to $(n - 1)(n)(n + 1)$. We also realize that the sum of the first n terms is simply the sum of the first $n - 1$ terms PLUS the nth term. That is,

$$n(n+1)(n+2) = a_1 + a_2 + \cdots + a_{n-1} + a_n = (a_1 + a_2 + \cdots + a_{n-1}) + a_n = (n-1)(n)(n+1) + a_n$$

Now we can rearrange the extreme left and right hand sides and write

$$a_n = n(n+1)(n+2) - (n-1)(n)(n+1) = n(n+1)(n+2-(n-1)) = n(n+1)(2+1) = 3n(n+1).$$

Now that we've done (c), we can do (b):

(b) $a_{18229} = 3(18229)(18230) = 996,944,010$.

6. The sum of fifty consecutive even integers is 3250. Determine the largest of the fifty integers.

Solution:

Let n be the smallest of these integers. Then we are given that

$$n + (n+2) + (n+4) + \cdots + (n+98) = 3250$$

Then

$$50n + 2 + 4 + \cdots + 98 = 3250$$

$$50n + \frac{1}{2}49(100) = 3250$$

(we get that $\frac{1}{2}49(100)$ by similar methods to that in question 3). So

$$50n = 3250 - 49(50) = 3250 - 2450 = 800$$

so

$$n = 16$$

and the largest is

$$n + 98 = 114.$$