Intermediate Math Circles
Number Theory I
Problems and Solutions

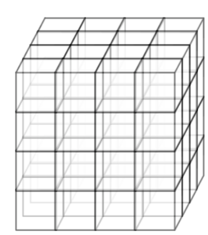
1. Complete your Sieve of Erastothenes for the values 151-200.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~
101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	~~115~~	~~116~~	~~117~~	~~118~~	~~119~~	~~120~~
~~121~~	~~122~~	~~123~~	~~124~~	~~125~~	~~126~~	127	~~128~~	~~129~~	~~130~~
131	~~132~~	~~133~~	~~134~~	~~135~~	~~136~~	137	~~138~~	139	~~140~~
~~141~~	~~142~~	~~143~~	~~144~~	~~145~~	~~146~~	~~147~~	~~148~~	149	~~150~~
151	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	157	~~158~~	~~159~~	~~160~~
~~161~~	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	~~169~~	~~170~~
~~171~~	~~172~~	173	~~174~~	~~175~~	~~176~~	~~177~~	~~178~~	179	~~180~~
181	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	~~187~~	~~188~~	~~189~~	~~190~~
191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199	~~200~~

2. Determine the prime factorization of the following numbers.
   (a) 364 (b) 693 (c) 3185

   (a) $364 = 2^2 \times 7 \times 13$ (b) $693 = 3^2 \times 7 \times 11$ (c) $3185 = 5 \times 7^2 \times 13$

3. A solid rectangular box is made of $1cm^3$ pieces. The volume of the rectangular prism is $1925cm^3$. How many different dimensions of the box are possible?



Since the volume of a rectangular box is found by $V = length \times width \times height$, the factors of 1925 can be used to determine all possible dimensions of this rectangular prism. It is important to note that the length, width and height of the rectangular prism must be positive integer values greater than 0.

The prime factorization of 1925 is $5^2 \times 7 \times 11$. The possible arrangements of these 4 prime factors to make the dimensions of the rectangular box are,

$$\begin{aligned} 5^2 \times 7 \times 11 &= 25 \times 7 \times 11 \\ 5 \times (5 \times 7) \times 11 &= 5 \times 35 \times 11 \\ 5 \times 7 \times (5 \times 11) &= 5 \times 7 \times 55 \\ 5 \times 5 \times (7 \times 11) &= 5 \times 5 \times 77 \end{aligned}$$

Therefore, there are 4 possible dimensions of the box.

4. The sum of the three digit number $2A3$ and 326 is $5T9$. If $5T9$ is divisible by 9, find the value of $A + T$.

   For the number $5T9$ to be divisible by 9, the sum of its digits must also be divisible by 9. i.e. $5 + T + 9 = T + 14$ must be divisible by 9. Since $0 \leq T \leq 9$, the only possible value of T is 4.

   Also, $A + 2 = T$, which means that $A + 2 = 4$. Therefore, $A = 2$.

   The sum $A + T = 2 + 4 = 6$.

$$\begin{array}{r} 2A3 \\ 326 \\ \hline 5T9 \end{array}$$

5. If n is a positive integer, then $n!$ is defined as the product of all integers from 1 to n inclusive. For example, $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6$. Determine the number of times that 5 will occur as a prime factor of $32!$.

   $32! = 1 \times 2 \times 3 \times \cdots \times 30 \times 31 \times 32$.
   Each number in this product that is a multiple of 5, will give a prime factor of 5. The multiples of 5 in this product are $5, 10, 15, 20, 25, 30$.
   This list could be written as $5, 2 \times 5, 3 \times 5, 4 \times 5, 5^2, 6 \times 5$.
   Therefore, the prime factor 5 will occur 7 times in $32!$.

6. Three adults are all younger than 40 years in age. The product of their ages is 17710. Determine the sum of their ages.

   The prime factorization of 17710 is $2 \times 5 \times 7 \times 11 \times 23$.
   These 5 prime factors will be used to make the product of the three ages. Since each adult is younger than 40, one adult must be 23 years old. To make the remaining two ages under 40, the 2 must be paired with the 11 and the 5 must be paired with the 7.
   This produces the product $23 \times (2 \times 11) \times (5 \times 7) = 23 \times 22 \times 35 = 17710$.
   Therefore the adults must be aged 22, 23, and 35.

7. The digits $1, 2, 3, 4, 5$ are each used only once to make a five digit number $abcde$. The number made by the three digits abc is divisible by 4. The number made by the three digits bcd is divisible by 5, and the number made by the three digits cde is divisible by 3. Determine the digit a.

   Since abc is divisible by 4, bc must be a multiple of 4. The possible combinations of the digits $1, 2, 3, 4, 5$ to make a multiple of 4 are $12, 24, 32, 52$.

   Since bcd is divisible by 5, $d = 5$ and none of a, b, c, e can be 5. The list of possible combinations of bc is shortened to $12, 24, 32$.

   Since cde is divisible by 3, then $c + d + e = c + 5 + e$ must also be divisible by 3. The list of possible combinations of cde is $153, 351, 354, 453$, which implies that c could only be 1,3 or 4. Examining the possible values of bc we can determine that $c = 4$ since the only possible value of bc that ends in 1,3 or 4 is 24. Consequently, cde must be 453.

   Therefore, $abcde$ must be $a2453$ and $a = 1$.

8. The product of 20^{50} and 50^{20} is written as an integer in expanded form. Determine the number of zeros at the end of the resulting integer.

   The number of zeros at the end of a positive integer is equal to the number of factors of 10 contained within the number. A factor of 10 is made by 2×5. To count the number of factors of 10 in the product $20^{50} \times 50^{20}$, count the number of pairs of prime factors 2 and 5.

$$\begin{aligned} 20^{50} \times 50^{20} &= (2^2 \times 5)^{50} \times (2 \times 5^2)^{20} \\ &= (2^2)^{50} \times (5)^{50} \times (2)^{20} \times (5^2)^{20} \\ &= (2)^{120} \times (5)^{90} \end{aligned}$$

   Therefore, there are 90 possibles pairings of 2 and 5 i.e. $(2 \times 5)^{90} = 10^{90}$ and the product of 20^{50} and 50^{20} will have 90 zeros.

9. The product of 792 and positive integer, n, is a perfect square. Determine the smallest possible value of n.

   The prime factorization of 792 is $2^3 \times 3^2 \times 11$. For the product of $792n$ to be a perfect square, each of its prime factors must be paired with itself. The number 792 has two pairs of equal factors, 2^2 and 3^2. The factors that must be paired to make $792n$ a perfect square are 2 and 11. Therefore, the smallest possible value of n is $2 \times 11 = 22$.

10. The number of divisors of 245, other than 1 and 245 are,
    (a) 3 (b) 4 (c) 5 (d) 6 (e) 7

    The prime factorization of 245 is 5×7^2.
    The divisors of 245 can be found with the following table.

	7^0	7^1	7^2
5^0	1	7	49
5^1	5	35	245

    The number 245 has 4 possible divisors, if 1 and 245 are not included.
    The answer is (b).

11. Determine the sum of the proper divisors of 1089.

    The prime factorization of 1089 is $3^2 \times 11^2$.
    The divisors of 245 can be found with the following table.

	3^0	3^1	3^2	Row Sum
11^0	1	3	9	13
11^1	11	33	99	143
11^2	121	363	1089	1573
Column Sum	133	399	1197	

$$\begin{aligned}\text{Sum of Divisors} &= \text{First Row Sum} \times \text{First Column Sum}\\ &= 13 \times 133\\ &= 1729\end{aligned}$$