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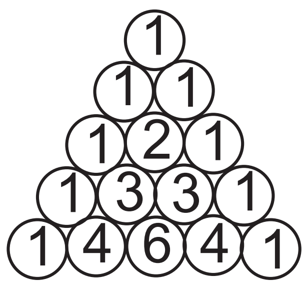
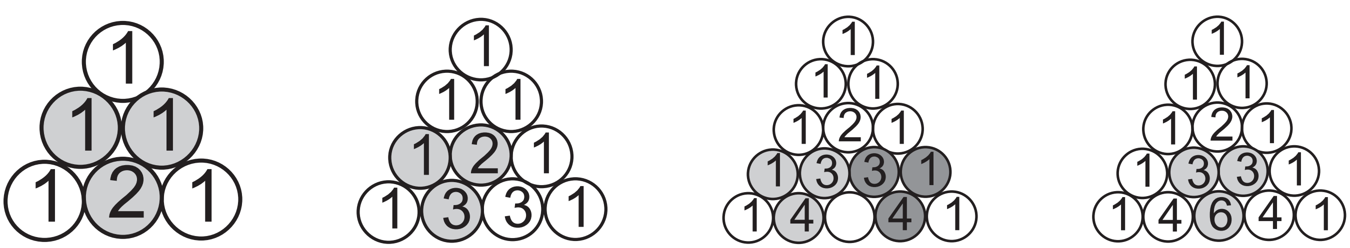
Grade 7/8 Math Circles
Winter 2013
Pascal's Triangle



Building the Triangle

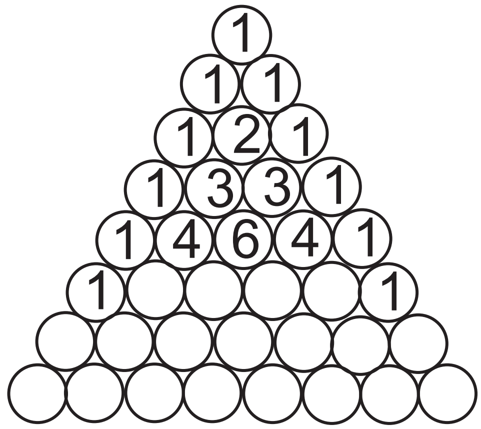
Pascal's Triangle is an interesting number pattern named after Blaise Pascal, a famous French mathematician. To build the triangle we start with 1 at the top, and continue adding numbers in a triangular shape.

All the edges of Pascal's Triangles are 1's, and each number in between is the sum of the two numbers above it.



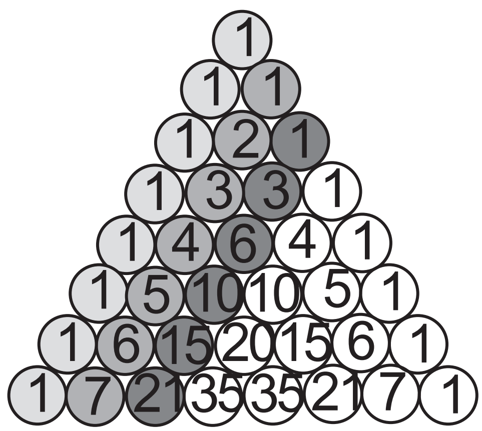
Exercise

Fill out the blanks in the Pascal Triangle below:



<u>Pascal Patterns</u>

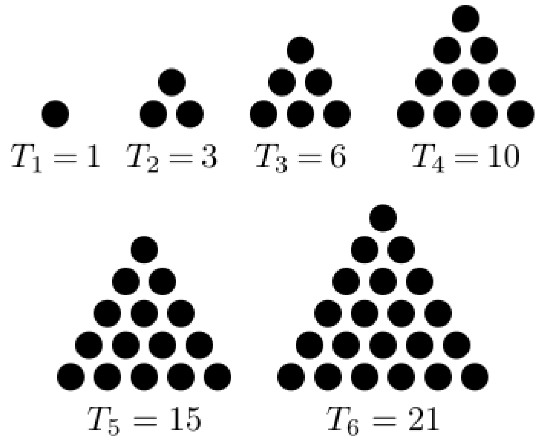
Diagonals



The first diagonal is just "1's".

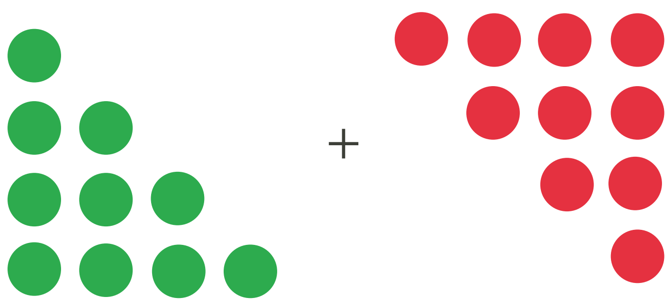
The second diagonal has the *counting numbers.*

The third diagonal has the *triangular numbers.* Triangular numbers, are the numbers of dots that it takes to make increasingly large triangles.

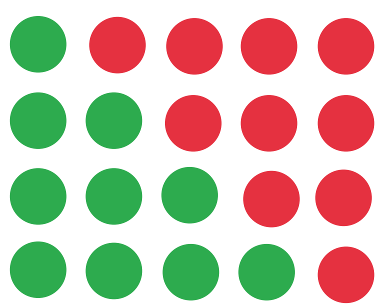


How do we get the next triangular number?

Let's say we want the 4th triangular number.



The two triangles of dots are triangles that corresponds to the 4th triangular number. If we put them together we get,



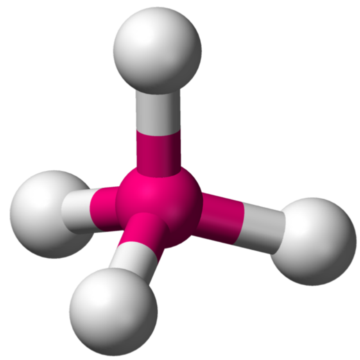
We now have a 4 x 5 rectangular. If we divide this by 2, we are cutting the rectangle into equal halves and get what we originally had.

That means to get the 4th triangular number we do: $\dfrac{4 \times 5}{2} = \dfrac{20}{2} = 10.$

Let 4 = n.

Then $\dfrac{n(n+1)}{2}$ = the nth triangular number.

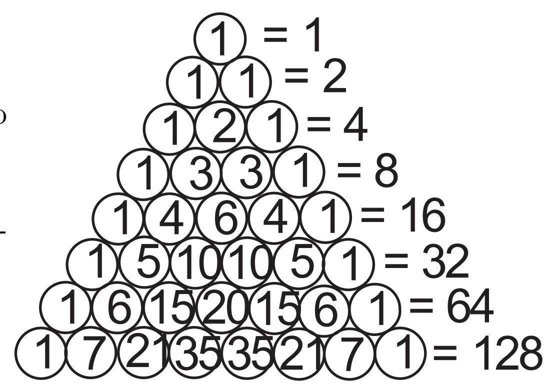
Back to Pascal's Triangle, the fourth diagonal has the tetrahedral numbers. Tetrahedral numbers are the number of dots it takes to make increasingly large 3D triangles. An example of the second tetrahedral number is below.



Horizontal Sums

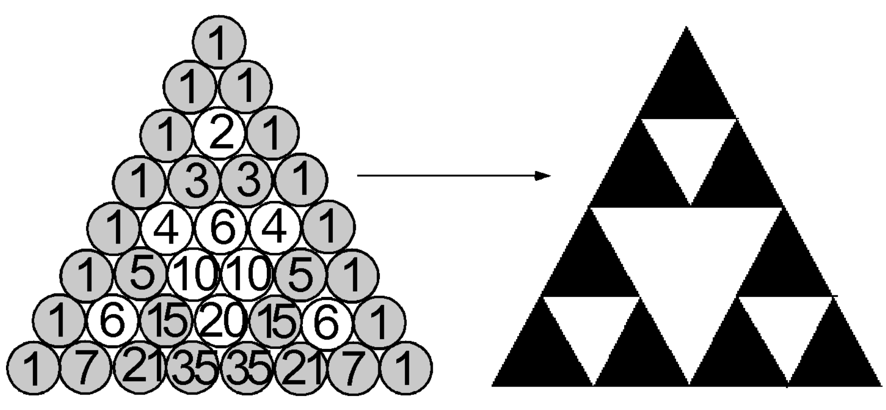
If we look at the rows of Pascal's Triangle we will also see another pattern. Can you see it?

The sum of each row doubles as you do down! The horizontal sums are the powers of 2!



Odds and Evens

Another pattern is formed when you colour in the odd and even numbers in Pascal's Triangle.

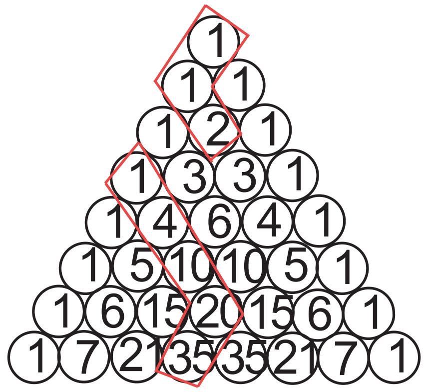


This triangle is *Sierpinski's Triangle.* Waclaw Sierpinski was a polish mathematician who uncovered this pattern of triangles in 1915.

Hockey Sticks

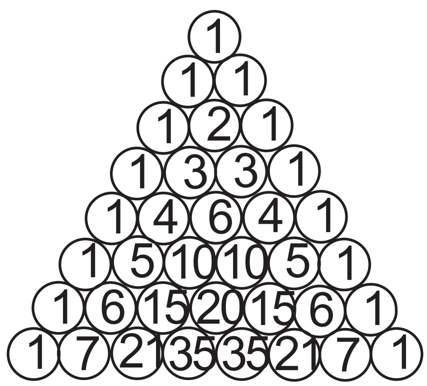
If you draw hockey stick shapes on Pascal's Triangle, all the numbers on the longer part of the stick will add up to the number on the shorter part of the stick.

Looking at the smallest hockey stick at the top we can see that: $1 + 1 = 2$ The biggest hockey stick at the bottom : $1 + 4 + 10 + 20 = 35$



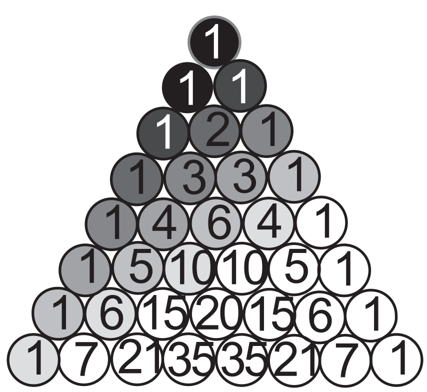
Exercise 1

Draw 3 more Hockey Sticks that work on Pascal's Triangle below:



Exercise 2

Add up the coloured circles in Pascal's Triangle. What do you see?



Fibonacci Series

Choose some 1 on Pascal's Triangle. By adding the numbers when going up one and over one, you get the Fibonacci Series!

Prime Numbers

If the first number of a row (excluding the 1) is a prime number, then all the numbers in that row are divisible by that number!

11th Exponent

Fill out the table below, and see if you can see any patterns between Pascal's Triangle and the powers of 11.

Power of 11	**Value**	**Row**
11^0	1	0
11^1	11	1
11^2	121	2
11^3	1331	3
11^4	14641	4
11^5	161051	5
11^6	1771561	6

As you can see, up until the power of 5, the powers of 11 match the numbers in Pascal's Triangle. But what about 11^5 and 11^6?

$11^5 = 161051 = 1\ (5+1)\ (0+1)\ 0\ 5\ 1 = 1\ 5\ 10\ 10\ 5\ 1$

The first half of the numbers (excluding 1) overlap to make 11^5! Try doing the same for 11^6.

$11^6 = 1771561 = 1\ (6+1)\ (5+2+0)\ 15\ 6\ 1 = 1\ 6\ 15\ 20\ 15\ 6\ 1$

Review

What is a factorial?

A factorial is a shorthand of writing the product of all integers from the given integer down to one.

We express 4 factorial by writing 4!

$4! = 4 \times 3 \times 2 \times 1$

What is $3! \times 2$? $(3 \times 2 \times 1) \times 2$

What is the choose function?

In probability, we use the choose function to find the number of possible outcomes from a total amount n, where we choose k items.

We express 3 choose 2 like $\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3 \times 2 \times 1}{(2 \times 1) \times (1)}$

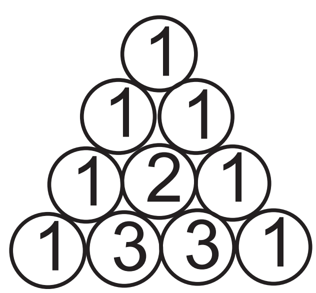
Formula for any entry

The formula for any entry in Pascal's Triangle from combinations, one of the applications of Pascal's Triangle. This formula is just the choose function, which we use to solve combination problems in probability.

The formula is: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Example

$$
\begin{matrix}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{matrix}
$$



Using Pascal's Triangle

Combinations

Pascal's Triangle is largely used to figure out combinations.

Exercise

Laura has 4 rubber balls in a bag: 1 red, 1 green, 1 blue, 1 yellow. How many ways can Laura pick 2 balls if order doesn't matter. Use Pascal's Triangle.

Solution:

Since Laura has 4 rubber balls, we go down to the 4th row. Ignoring the one, we go to the second spot of row 4. The value is 6. So Laura can pick 2 balls out of his 4 in 6 ways.

If you use the choose formula, n = 4 (the total amount of balls) and k = 2 (the amount you are choosing): $\binom{4}{2} = 6$

Which is also the formula for the 4th row, 2nd spot in Pascal's Triangle. Cool!

Probability: Heads and Tails

Let's see how Pascal's Triangle can be used when tossing a coin.

Tosses	**Possible Outcomes**	**Pascal's Triangle**
1	H T	1 1
2	HH HT TH TT	1 2 1
3	HHH HHT HTH THH HTT THT TTH TTT	1 3 3 1

Try 4 tosses on your own.

Solution:

HHHH
HHHT HHTH HTHH THHH
HHTT HTHT HTTH THHT THTH TTHH
TTTH TTHT THTT HTTT
TTTT

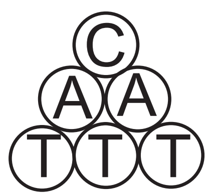
Pascal's Triangle: 1 4 6 4 1

Notice when you add up the numbers in Pascal's Triangle for each toss, you can the total number of outcomes.

$$1 \text{ Toss} = 2 = 1 + 1$$
$$2 \text{ Toss} = 4 = 1 + 2 + 1$$
$$3 \text{ Toss} = 8 = 1 + 3 + 3 + 1$$
$$4 \text{ Toss} = 16 = 1 + 4 + 6 + 4 + 1$$

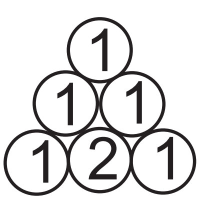
Exercises

A. In how many ways can you write the word CAT starting at C and working down?



Solution:

You can write CAT in 4 ways, starting at the top and working to the bottom row. But let's put Pascal's Triangle in the circles instead of CAT. We get,

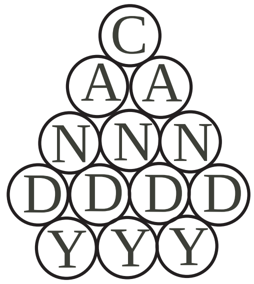


The sum of the bottom row is 1 + 2 + 1 = 4. Which is the same number of ways you can write CAT. Why?

Because the number of ways you can write CAT with the first T at the end is 1. The number of ways you can write CAT with the 2nd T at the end is 2. And the number of ways you can write CAT with the 3rd T at the end is 1. So in total the number of ways you can write CAT with all the T's is 4.

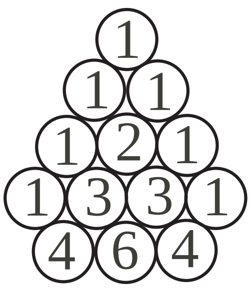
In other words, what we are really counting is how many ways can we get to each T in CAT. Adding up all the ways we can get to each T, we get all the ways we can spell CAT.

B. In how many ways can you write the word CANDY starting at C and working down?



Solution:

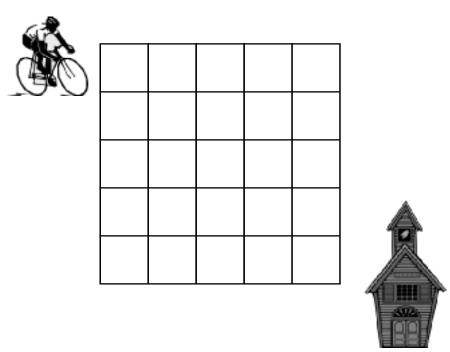
You can write CANDY in 14 ways. Like in the previous example, we can put Pascal's Triangle in the circles to get,



Again, the number of ways to get to the first Y is 4. The number of ways to get to the second Y is 6. And the number of ways to get to the third Y is 4. So, we can spell CANDY with the fist Y in 4 ways, 6 ways with the second Y, and 4 ways with the third Y.

In total, we have $4 + 6 + 4 = 14$ ways to write CANDY.

C. Johnny rides his bike to school every day. But the bike routes only run south and east. How many different routes can Johnny take to get to school?



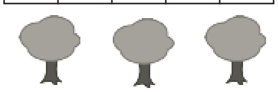
Solution:

The number inside each box is the number of possible ways Johnny can ride his bike to that box. It follows Pascal's triangle, because the number of ways to get to any box is the number of ways the box to the sum of the box above it and to the left of it. Why? Because the bike routes only run south and east, so each box it can only be reached by the box above or to the left of it.

1	1	1	1	1
1	2	3	4	5
1	3	6	10	15
1	4	10	20	35
1	5	15	35	70

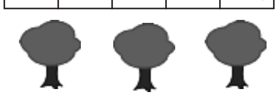
The number at the box by the school is 70, so the number of ways Johnny can get to school everyday is 70.

D. Johnny also likes to ride his bike to the park. The park however, isn't located any one address. How many ways can Johnny get to the park?



Solution:

1	1	1	1	1
1	2	3	4	5
1	3	6	10	15
1	4	10	20	35
1	5	15	35	70



Since the park isn't at any one address, that means it isn't matched with just one box. The entire bottom row is a way to get to the park. So the different place Johnny can get to at the park are the boxes with 1, 5, 15, 35, 70.

So to get all the possible ways he can get to the park we add $1 + 5 + 15 + 35 + 70 = 126$

Problem Set

1. Write out the 9th row of Pascal's Triangle?

2. How many way can you spell CAKE starting at C? Use Pascal's Triangle.

3. Using Pascal's Triangle, how many combinations give a sum of 56?

4. Continue Sierpinski's triangle for one more row of triangles. What numbers would be in the white triangles?

5. Carlos Pizza Shop offers 9 kinds of pizza toppings: onions, tomatoes, peppers, bacon, ham, extra cheese, spinach, pepperoni, olives.

   a) How many kinds of pizza can someone order if they can only choose 1 topping?

   b) How many kinds of pizza can someone order if they can choose 8 toppings?

   c) How many kinds of pizza can someone order if they can choose 3 toppings?

   d) How many kinds of pizza can someone order if they can choose 7 toppings?

   e) What do you notice about these values?

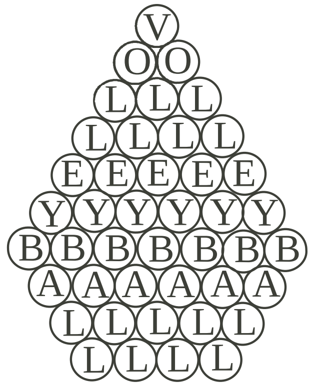
   f) Without doing any kind of calculating, how many kinds of pizza can someone order if they can choose 2, 4, 5, or 6 toppings?

6. What is the sum of row 43?

7. What is the value of 6! ?

8. Sebastian has a box of 30 jelly beans. His sister came and ate 20 of his jelly beans! In how many ways can Sebastian eat the rest of his jelly beans if he can only fit 4 in his mouth at a time, and order doesn't matter?

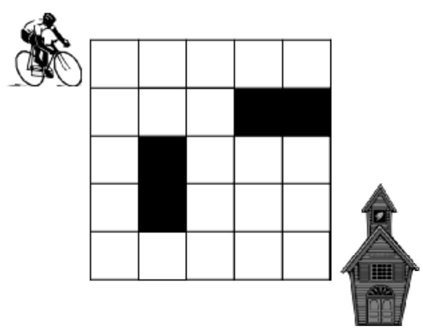
9. How many ways can you spell Volleyball starting at V?



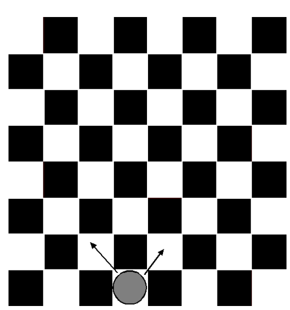
10. What is the first number of the row that has the numbers 78 286 715 1287 1716 1716 1287 715 286 78?

11. Show how 11^7 is the same as the 7th row of Pascal's Triangle.

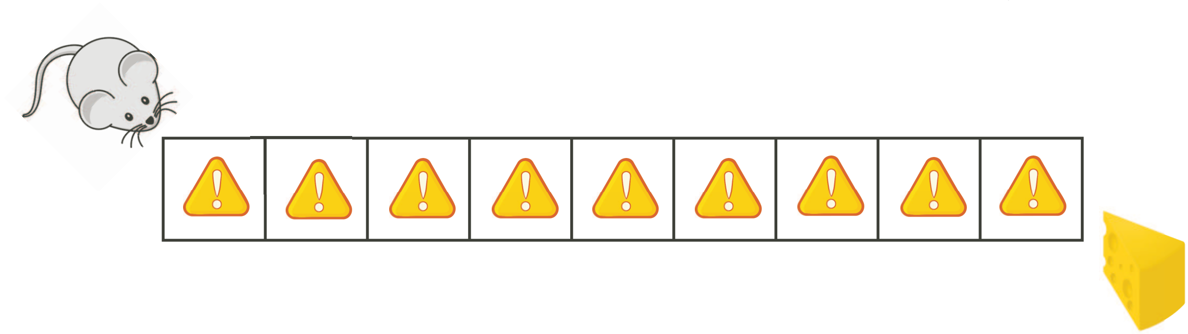
12. There is construction on the road and some of Johnny's bike routes are blocked. How many different routes can Johnny take to get to school now?



13. In checkers, players are allowed to move only in diagonals from their starting position. How many moves can the grey checkers piece move from its current place? Keep in mind you can not move backwards.



14. Madam Souris wants to get the cheese located on the opposite side of the table. But in every box there is a mouse trap that Madam Souris must avoid. She can walk in any direction, but she can not retrace her steps. Also, the distance of every line segment is 1 cm. If she needs to run a distance of at least 14 cm, how many ways can she reach the cheese? (Hint: Think of the 10th row of Pascal's Triangle: 1 10 45 120 210 252 210 120 45 10 1)

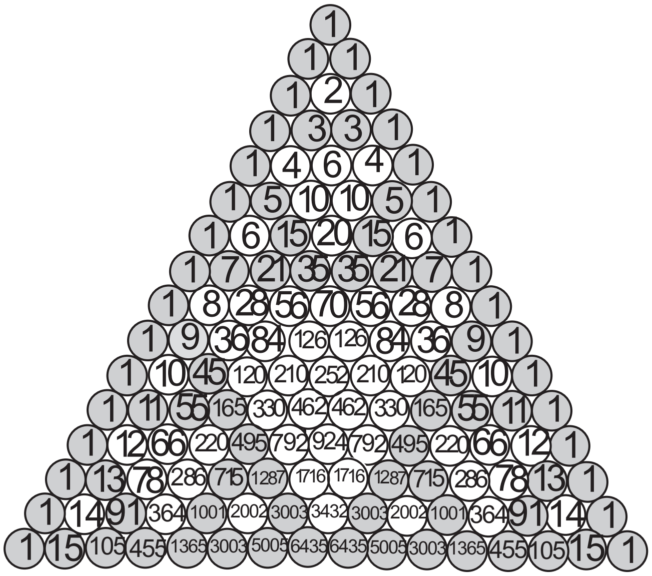


Solutions

1. 1 9 36 84 120 120 84 36 9 1

2. 8 ways

3. There are two ways to have a sum 56: 21 + 15 + 10 + 6 + 3 + 1 = 56, and 35 + 15 + 5 + 1 = 56, but you can draw 4 hockey sticks.

4. 

5. a) 9
   b) 9
   c) 84
   d) 36
   e) a) is the first number of the 9th row, b) is the 8th number, c) is the 3rd number, d) is the 7th number
   f) 2 toppings - 36 kinds, 4 toppings - 120 kinds, 5 toppings - 120 kinds, 6 toppings - 84 kinds

6. 2^{43} = 8,796,093,022,208

7. 720

8. Assume Sebastian wants to eat as many as he can each time. First he can eat $\binom{10}{4}$. Then $\binom{6}{4}$, and lastly $\binom{2}{2}$. So all together he can eat the rest of his jelly beans in 226 ways.

9. 420

10. 13

11. 11^7 = 19487171, and the 7th row of Pascal's Triangle is 1 7 21 35 35 21 7 1.
    1 + (7 + 2) + (1 + 3) + (5 + 3) + (5 + 2) + 1 + 7 + 1

12. 19 different routes
13. 103 ways
14. Consider the number of ways Madam Souris can go up and down.

$$\binom{10}{5} + \binom{10}{7} + \binom{10}{9} = 252 + 120 + 10 = 382 \text{ ways}$$