University of Waterloo
Faculty of Mathematics

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Grade 7/8 Math Circles
Winter 2013
3D Geometry

Introductory Problem

Mary's mom bought a box of 60 cookies for Mary to bring to school. Mary decides to bring 30 cookies to school. In how many ways can Mary stack the cookies(on top of one another and/or side by side) to make a box-like figure so it will be easy for her to bring them to school?

Solution:

To find the different ways Mary can stack the cookies, we need to consider the factors of 30.

The factors of 30 are: 1, 2, 3, 5, 6, 10, 15, 30

Using these factors, how should we group the factors up so Mary can make a box-like figure with her cookies? (Hint: consider how many dimensions a box has?)

A box has three dimensions: length, width and height

So, we need to group the factors in groups of three. These groups are:
(1,1,30),(1,2,15),(1,3,10),(1,5,6),(2,3,5)

So there 5 ways Mary can stack the cookies to make a box-like figure.

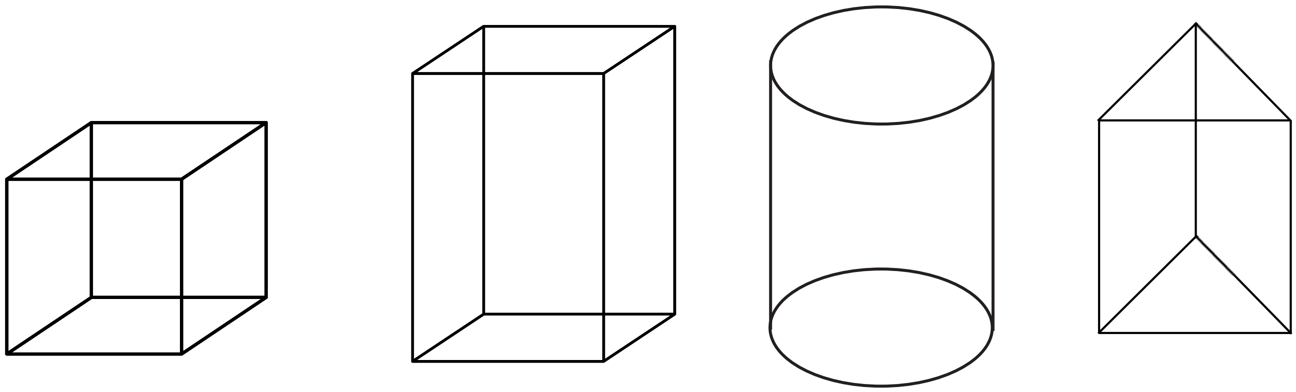
Volume of 3D Figures

The *volume* of a 3D figure is the amount of space within the 3D figure. We measure the volume of any 3D figure in cubic units.

Prisms

Prisms are 3D figures that have congruent parallelogram sides, and a solid base, which is either of two parallel ends on the figure.

Examples



Each figure above is a kind of prism. The first is called a cube. The second is called a rectangular prism. The third is a cylinder, and the fourth is a triangular prism.

The formula to find the volume of a prism is:

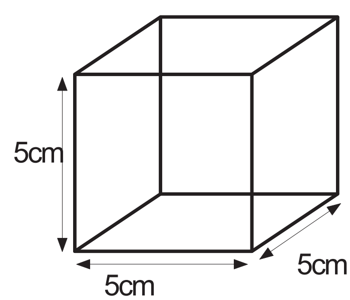
$$V = \text{area of base} \times \text{height}$$

Jack in the Box

Jack's box is 5 cm wide, 5 cm long, and 5 cm tall. How much room does Jack have inside his box?

Solution:

Jack's box is a cube, which means that the base of this prism will be a square. Using the formula:



$$\begin{aligned}\text{Area of base} &= 5 \times 5 \\ &= 25 \text{ cm}^2 \\ \text{Height} &= 5 \text{ cm} \\ V &= 25 \times 5 \\ &= 125 \text{ cm}^3\end{aligned}$$

Hence, the volume of Jack's box is 125 cm^3.

Kaleidoscope

A kaleidoscope is a cylinder that contains a triangular prism inside made up of mirrors. Inside triangular prism are colorful beads and small pieces of glass. Light reflects off the mirrors of the triangular prism, the beads and the glass so that when a person looks through they see many colors and patterns.

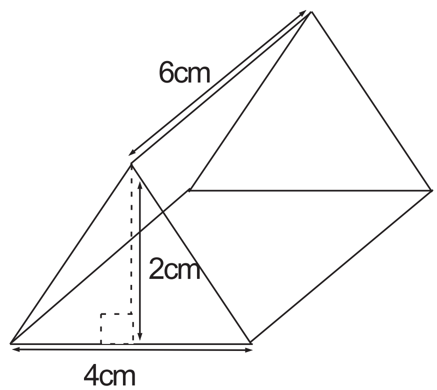
a) If the dimensions of the triangular prism of a kaleidoscope a height of 2 cm, a base of 4 cm, and a length of 6 cm, what is the maximum number of beads and glass that can fit in the prism?

Solution:

$$\text{Area of base} = \frac{bh}{2} = \frac{(4)(2)}{2} = 4\ \text{cm}^2$$

Length = 6 cm

$$V = \frac{bh}{2} \times \text{h} = 4 \times 6 = 24\ \text{cm}^3$$



b) What would the height of the cylinder containing the triangular prism have to be? What would be the radius?

Solution:

The height of the triangular prism is 6 cm, so height of the cylinder needs to be at least 6 cm.

Since the base of the triangle of the triangular prism is 4 cm, the diameter of the cylinder needs to be at least 4 cm. Meaning the radius needs to be at least 2 cm.

So the dimensions of the cylinder would be a height of at least 6 cm and a radius of 2 cm.

c) With all the information you have, what is the volume of the cylinder of this kaleidoscope?

Solution:

$$\begin{aligned}\text{Area of base} &= \pi\ \mathrm{r}^2\\ &= \pi\ 2^2\\ &= 12.56\ \mathrm{cm}^2\end{aligned}$$

Height = 6 cm

$$\begin{aligned}V &= \pi\ \mathrm{r}^2\ \mathrm{h}\\ &= 12.56 \times 6\\ &= 75.36\ \mathrm{cm}^3\end{aligned}$$

d) How much space is there between the cylinder and the triangular prism?

Solution :

Volume of cylinder = 75.36 cm^3
Volume of triangular prism = 24 cm^3

75.36 - 24 = 51.36

Hence the amount of space between the cylinder and the triangular prism is 51.36 cm^3

Cones and Pyramids

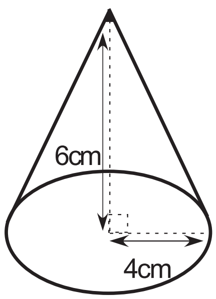
A *cone* is a 3D figure that have a circular base and a rectangular face that wraps around the circumference of the base into a point, called a *common vertex*.

$$\text{The formula for a cone is: } V = \frac{1}{3}\pi\ \mathrm{r}^2\ \mathrm{h}$$

A *pyramid* is a 3D figure that has a polygon base, and all other faces that are triangles and meet at a common vertex.

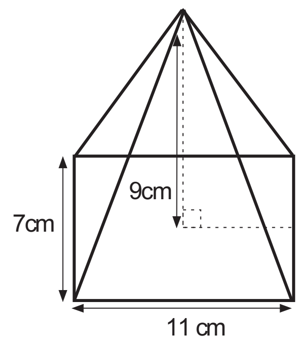
$$\text{The formula for a pyramid is: } V = \frac{1}{3}\times \text{ area of base} \times \text{height}$$

Exercises

1) 

Solution:

$$
\begin{aligned}
V &= \frac{1}{3}\pi\ \mathrm{r}^2\ \mathrm{h} \\
&= \frac{1}{3}\pi\ (4^2)\ \mathrm{x}\ 6\ \mathrm{cm} \\
&= \frac{1}{3}\times\ (50.24\ \mathrm{cm}^2\ \mathrm{x}\ 6\ \mathrm{cm}) \\
&= \frac{1}{3}\times\ 301.44\ \mathrm{cm}^3 \\
&= 100.44\ \mathrm{cm}^3
\end{aligned}
$$

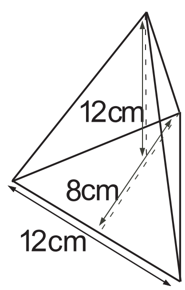
2) 

Solution:

$$
\begin{aligned}
\text{Area of base} &= l \times w \\
&= 7 \times 11 \\
&= 77\ \mathrm{cm}^2
\end{aligned}
$$

Height $= 9$ cm

$$
\begin{aligned}
V &= \frac{1}{3}\times\ (l \times w \times h) \\
&= \frac{1}{3} \times 77 \times 9 \\
&= 231\ \mathrm{cm}^3
\end{aligned}
$$

3) 

Solution:

$$
\begin{aligned}
\text{Area of base} &= \frac{bh}{2} \\
&= \frac{(12)(8)}{2} \\
&= 48\ \mathrm{cm}^2
\end{aligned}
$$

Height $= 12$ cm

$$
\begin{aligned}
V &= \frac{1}{3} \times \frac{bh}{2} \times \mathrm{h} \\
&= \frac{1}{3} \times 48 \times 12 \\
&= \frac{1}{3} \times 576\ \mathrm{cm}^3 \\
&= 192\ \mathrm{cm}^3
\end{aligned}
$$

Spheres

A *sphere* is a 3D figure whose surface is at all points equally distant from the center. This distance from the center of the sphere to the surface is called the *radius*.

The formula for the volume of a sphere is: $V = \frac{4}{3}\pi \mathrm{r}^3$

Example

8 soccer balls at Lucy and Eric's school have no air. For being late to class, Lucy and Eric's teacher told them to fill up the 8 soccer balls with air. If the radius of each soccer ball is 12 cm, how much air will they need to fill up the balls?

Solution:

We know the radius of one soccer ball is 12 cm. So,

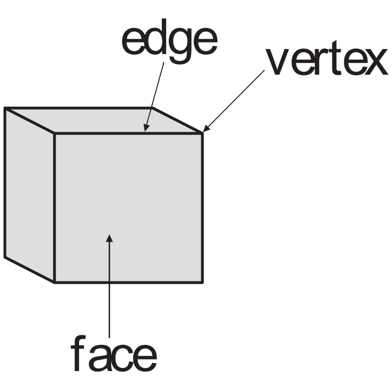
$$V = \frac{4}{3}\pi 12^3 \text{ cm}$$
$$= 7234.56 \text{ cm}^3$$

So the volume of one soccer ball is 7234.56 cm^3.

This is equivalent to needing 7234.56 cm^3 of air for one ball. So for 8 balls the amount of air needed is: 7234.56^3 x 8 = 57876.48 cm^3.

<u>Euler's Formula</u>

Definitions:



A *face* is any of the shape surfaces of a 3D figure.

A *polyhedron* is a 3D figure having many faces.

An *edge* is a line segment connecting two faces on polyhedrons.

A *vertex* is where three or more edges meet.

Euler was a famous mathematician who discovered a relationship between the faces, edges and vertices of polyhedron. His equation works for all polyhedrons which do not intersect with themselves at some point. We call these kinds of polyhedrons *convex*.

Exercise

Fill out the table below and try to find the relationship between the faces, vertices and edges of polyhedrons.

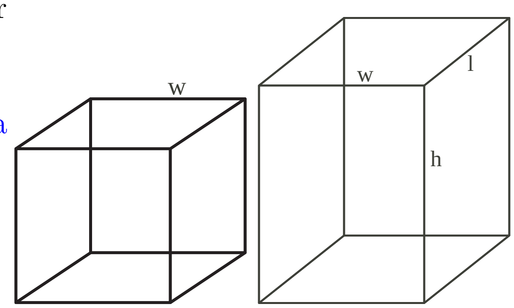
Names	Faces	Edges	Vertices
Cube	6	12	8
Rectangular Prism	6	12	8
Triangular Prism	5	9	6
Square Based Pyramid	5	8	5
Pentagonal Prism	7	15	10
Tetrahedron	4	6	4
Octahedron	8	12	6
Dodecahedron	12	30	20

Euler's Formula is: F + V = E + 2

Surface Area

The *surface area* of a prism is the area of all the combines surfaces of the prism. This is true for pyramids and cones as well.

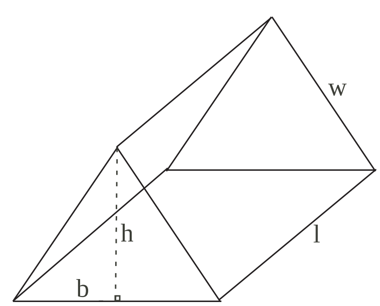
How would you find the surface area of a cube or rectangular prism?



A cube or rectangular prism has 6 faces. So we add up the area of all 6 sides to get the surface area.

$\text{SA}_{cube} = 6w^2$
$\text{SA}_{rect.} = 2lw + 2wh + 2lh$

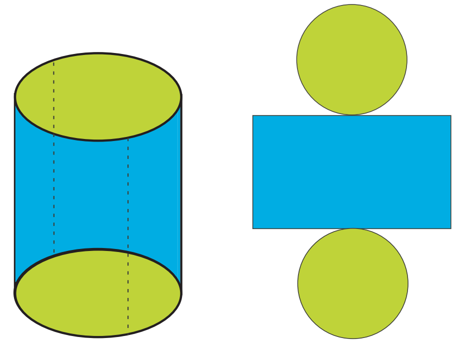
How would you find the surface area of a triangular prism?



A triangular prism has 5 faces. So we add up the area of all 5 sides to get the surface area.

$\text{SA} = bh + 2lw$

How would you find the surface area of a cylinder?



A cylinder has 2 identifiable circular faces and one surface that wraps around the two sides. This surface is actually a curled rectangular, with a length of the circumference of a circle, and a width of the height of the cylinder. We add up the area of the 2 circular sides, and the rectangle to get the surface area.

$\text{SA} = 2\pi r^2 + (\pi d \times h)$

Problem Set

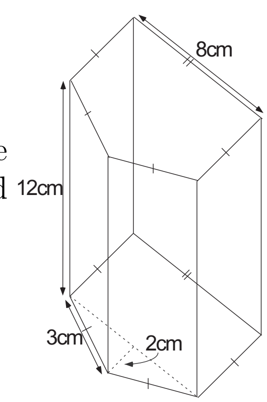
1. The Great Pyramids of Giza are named the Pyramid of Menkaure, the Pyramid of Khafre and the Pyramid of Khufu. The height and base of each of these pyramids are: 65.5 m by 103.4 m^2, 136.4 m by 215.25 m^2, 138.8 m by 230.4 m^2 respectively. What is the volume of each of the Great Pyramids of Giza?

2. Knowing that an icosahedron has 12 vertices and 30 edges, how many faces does it have?

3. Mark wants to build himself a large compost bin for his farm. He measured that he typically has 100m^3 of compost a week. Name all the possible combinations of dimensions that he can build his garbage can if the can will be in the shape of a rectangle?

4. Forty-two cubes with 2 cm edges are glued together to form a rectangular prism. If the area of the base of the prism is 24 cm and the width of the base is greater than 2 cm, what is the height of the prism?

5. Daisy bought herself a vase to fill with potpourri for Christmas. If the vase has the dimensions as shown, how much potpourri will she need to fill the vase up to the top?



6. Find the smallest cylinder than can fit a cube of 1000 m^3.

7. A business downtown keeps erasers in boxes with dimensions 24 cm x 28 cm x 13 cm. The erasers have dimensions 2 cm x 4 cm x 1 cm. One of the boxes is half full with erasers.

   a) How much room is left in the box to put more erasers?

   b) How many erasers can fit into the empty half of the box?

8. Looking back at the question about the Great Pyramids of Giza, how much space to the three pyramids take up on the Giza plateau altogether?

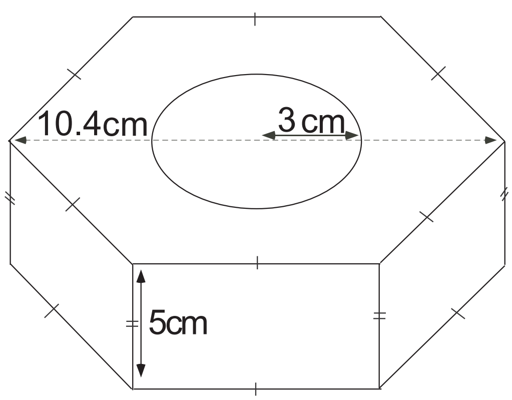
9. Calculate the amount of metal needed to make 8 cylindrical cans with a diameter of 6 cm and a height of 16 cm.

10. Dean is building a swimming pool in his backyard. The swimming pool will be 18 m long, 24 m wide, and 4.5 m deep. The pool is going to be tiled, with a tile size if 1 m^2, and it will cost 15 cents per square meter.

    a) What will it cost to tile the pool?

    b) How much water can the pool hold?

11. A bolt has a hexagon-like shape head. What is the volume of the head of a bolt with dimensions as shown? What is the surface area?

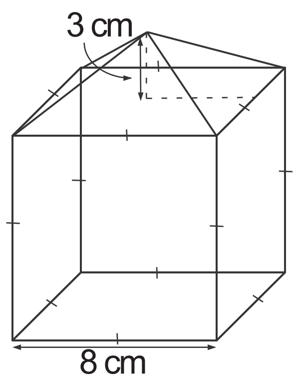


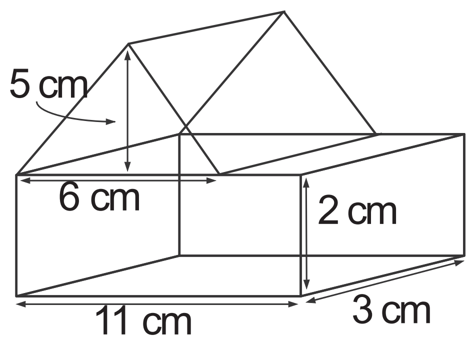
12. A enormous triangular prism holds a triangular shaped chocolate bar. How much cardboard is needed to create a casing that has a base of 12 cm, a height of 8 cm and a length of 20 cm?

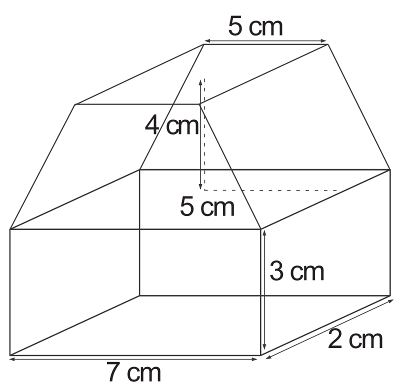
13. A hemisphere is half of a sphere. If the radius the hemisphere is 10 mm, what is the volume of the hemisphere?

14. A water pipe section with a volume of 50 m^3 and a diameter of 5 m burst. How long should the sheet metal be to create an identical piece of pipe?

15. Find the volume of the following figures:

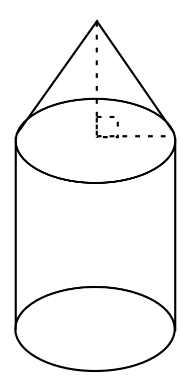
a) 

b) 

c) 

16. The volume of a sphere is 70 cm^3. What is the radius?

17. A small water bottle can hold 389.36 ml of water. Assuming for simplicity, the shape of a typical water bottle is a cylinder with a cone on top, as shown, with a radius of 5 cm. If the total height of the bottle is 6.37 cm, and the height of the cone is half the height of the cylinder, what is the height of the cylinder, and what is the height of the cone?



18. A wedge is a right triangular prism used to prop open doors. Assume a giant wedge is 12 cm long and 8 cm high, with a depth of 8 cm, and is propping a giant door open. If the space between the bottom of a door and the floor is 2 cm, what percentage of the wedge would be under the door?

19. A new tablet is formed through attaching two hemispheres to the ends of a cylinder with a height of 610 mm and radius r. If the volume of the tablet is equal to the volume of a cone of height 189 cm and radius r, find the value of r in mm.

Solutions

1. Menkaure: 2257.57 m^3; Khafre: 9786.7 m^3; Khufu: 10659.84 m^3
2. 20 faces
3. Possible sets are (1,2,50), (1,4,25), (1,5,20), (1,10,10), (2,2,25), (2,5,10), (4,5,5).
4. 14 cm
5. 384 cm^3 of potpourri
6. The smallest cylinder that can fit into a cube of 1000 m^3 has height of 10 cm and a radius of about 6 cm.
7. a) 4368 cm^3
   b) 546 erasers
8. 549.05 m^2
9. Area of the circles is $2 \times 3^2\pi$ cm^2 = 56.52 cm^2; Area of rectangle is 96π cm^2; Total surface area is 56.52 cm^2 + 301.44 cm^2 = 357.96 cm^2; 357.96 × 8 = 2863.68 cm^2
10. a) $ 121.5; b) 1944 m^3
11. The volume is 241.2 cm^3. The surface area is 182.28 cm^2.
12. 961.2 cm^2
13. 2093.33 mm^3
14. 2.55 m
15. a) 576 cm^3; b) 111 cm^3 ; c) 90 cm^3
16. $r^3 = 16.72$, so $r = \sqrt[3]{16.72} = 2.56$
17. h = 4.25 cm; height of cylinder = 4.25 cm; height of cone = 2.13 cm
18. 6.25 %
19. The volume of the tablet is $\frac{4}{3}\pi r^3 + 610\pi r^2$. The volume of the cone is $\frac{1}{3}1890\pi r^2$. These two volumes are equal, so $\frac{4}{3}\pi r^3 + 610\pi r^2 = 630\pi r^2$. Solve for r. r = 15 mm