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Grade 7 & 8 Math Circles
Circles, Circles, Circles

March 19/20, 2013

Introduction

The circle is a very important shape. In fact of all shapes, the circle is one of the two most useful shapes that exist (the other being the triangle)! But why is this? Well, circles have so many unique properties which can be used to solve many problems. Today we will focus on *some* of these properties and *some* of the applications.

Review

You have all worked with circles before. Let's review some properties of circles that you should already know.

The **circumference** is the closed curved line which defines the boundary of the circle. It's similar to the *perimeter* of other polygons. We can use the letter C in equations.

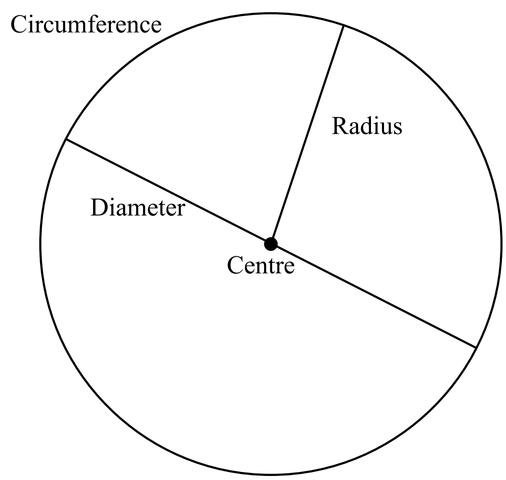
The **centre** of a circle is the point which is *equidistant* (meaning equally as far) from all points on the circumference. When drawing a circle with a **compass**, one leg will rest on the centre of the circle, while the the other (usually with a pencil tip) will draw the circumference.

The **radius** of a circle is the straight line distance from the centre to *any* point on the circumference. Remember, since the centre is *equidistant* from all points on the circumference, the radius will be the same no matter which point you choose on the circumference.

In equations, we denote radius with r.

The **diameter** of a circle is the length of a straight line which passes through the centre and has endpoints on the circumference. The diameter is also twice the length of the radius. Diameter is shown by d.

The diagram below summarizes all this information.



There are also a few equations which relate to properties of a circle. The length of the circumference is related to the radius. $C = 2\pi r$. Remember that the diameter is also twice the value of the radius. So we also have the equation $C = \pi d$.

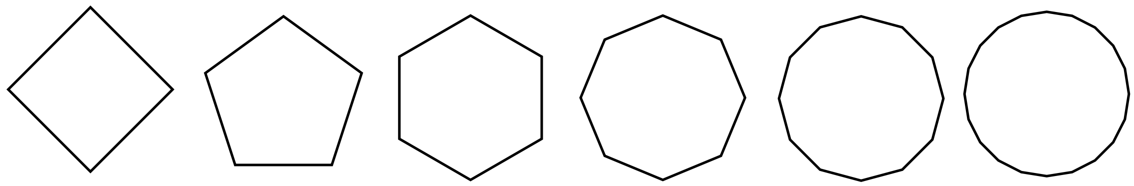
The equation for the area of a circle is $A = \pi r^2$

Recall that $\pi = 3.14159265...$ but we can use 3.14159 or even 3.14 to get a close enough estimate.

Now that we've reviewed some basic information, let's use these ideas to learn more about circles.

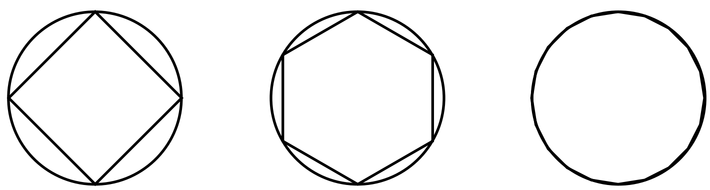
Tangent Line

Let's look at some other polygons for a while. Each polygon is **regular** meaning all the angles and side lengths are equal.



4 Sides 5 Sides 6 Sides 8 Sides 12 Sides 20 Sides

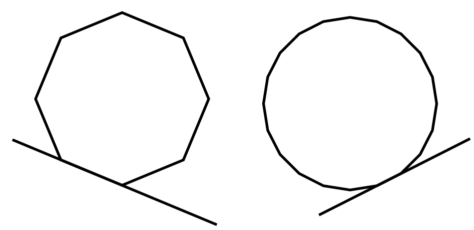
Notice how the more sides a regular polygon has, the more it looks like a circle. In fact, the more sides there are the closer the area comes the that of a circle. If we draw a circle around each polygon with the vertices on the circumference, this fact also becomes obvious. When we do this we say that the polygons are **inscribed** in a circle. Here are three polygons inscribed in circle.



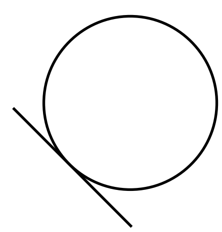
4 Sides 6 Sides 20 Sides

So we can think of a circle as a polygon with infinitely many short sides!

Now let's return to our polygons; specifically an octagon and an icosagon. Take one side of each figure and extend it on both ends like so:



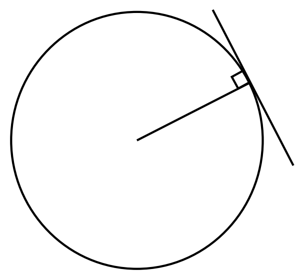
Since we can think of a circle as having infinitely many short sides, we can also extend one side:



Notice how on the polygons, the line only touches the one side and two vertices; it never intersects with any other side. The same thing happens with a circle. However since the side length is so small, we say this line only touches one point on the circle. A long line which intersects the circumference exactly once is called a **tangent line**.

Here's a real world example of tangents: Take any cylindrical object, paper towel roll, a pencil or a coin, and set it on it's side on a desk. Position your head so that you're looking directly across the table. You should see the desk creates a tangent line to the circular part of the cylinder.

Now for an important theorem involving tangent lines: The line drawn perpendicular to any radius line at the endpoint of the radius is the tangent line at that point on the circumference. Here's an example:



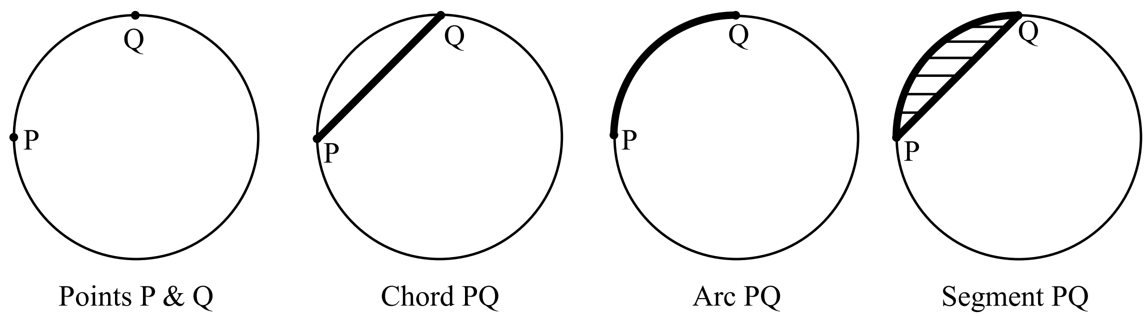
More Terminology

Choose any two points on the circumference of a circle and join them with a line. This line is called a **chord**. We've already worked with one specific chord; the diameter. So we can define the **diameter** more precisely as a chord which passes though the centre of the circle. The longest chord in every circle is the diameter.

Similarly, if we take any two points on a circle but this time connect them by using a section of the circumference as opposed to a straight line, we get an **arc**. Note there are actually two arcs between any two points; one goes clockwise and the other counter-clockwise from one point until it reaches the second point. In general, if referring to an arc between two points, the shorter arc is usually implied.

One arc you've used is the circumference. It begins and ends at the same point, so this is a special case!

Now if we take the arc and the chord between two points we get an enclosed region. This region is called a **segment**. Since there are two possible arcs between any pair of points, there are also two segments of any points. Again, we usually assume to use the smaller arc.



Finally, the area enclosed by 2 radii (plural of radius) and an arc between the radii is called a **sector**. This resembles a piece of pizza or a slice of pie! Who knew math could be so tasty?

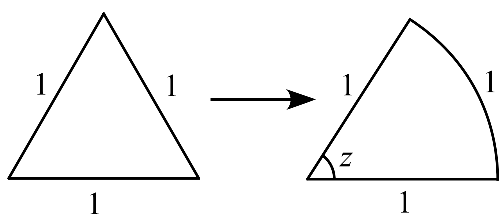
The length of an arc or the area of a segment can sometimes be calculated by using fractions of a circle. Let's take and example:

Circle, C_1, has a radius of 10cm. 5 radii are drawn around the circle, with angles equal between adjacent radii. What is the length of the arc and area of the sector between any two adjacent radii? Be sure to include a diagram in your solution.

Angles

Though using fractions can be nice when calculating arc length or area of a sector, most times there won't be a nice fraction. Instead we can use the angle separating two radii which work out nicely with more numbers. However, even when we use degrees many numbers won't work out nicely. This is why mathematicians don't like to use degrees but a more convenient units called **radians**.

But what is a radian? Let's begin by drawing an equilateral triangle with side lengths of 1 unit. Now take the one edge and "pull" it so the line is curved but still has a length of 1 unit. You'll get something like this:

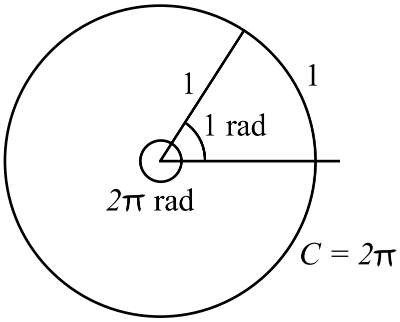


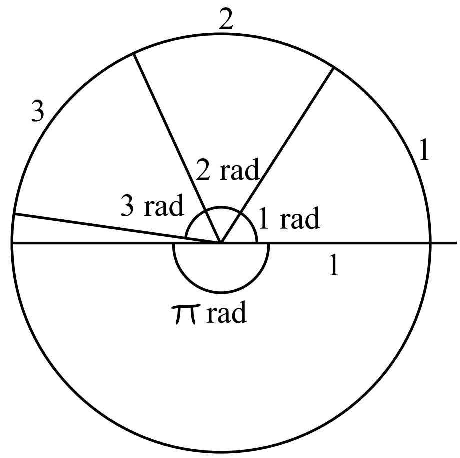
What part of a circle is this?

____

With similar triangles, all angles will be the same and the proportions of the side lengths will remain the same though the actual lengths may differ. The same can be said for similar sectors. So if we take the above sector, a similar sector will also have the same proportions between side lengths, though the lengths themselves may differ, and the same angle z. We define 1 **radian** or 1 rad to be the size of angle z.

Now let's extend this concept. How many radians are in a circle? First notice that the length of the arc is 1 unit, as is the radius of the circle which this sector was taken from. If we calculate the circumference using $C = 2\pi r$, we get $C = 2\pi$. In this specific circle, there is a direct correlation between arc length and angle (in radians). So in a circle there are 360° or 2π radians. The following diagrams should help explain further.





When converting angles from degrees to radians, we have $R = D \times \pi \div 180$. We can also convert from radians to degrees by using $D = R \times 180 \div \pi$, where D is an angle in degrees and R is an angle in radians.

Convert the following angles to radians or degrees.

1. 180°= ______________
2. $\frac{\pi}{2}$ rad = ______________
3. 45°= ______________
4. 1 rad = ______________
5. 270°= ______________
6. $\frac{3\pi}{4}$ rad = ______________

Arc Length & Sector Area

When an angle in radians is left as a fraction rather than a decimal, calculating arc length becomes very simple. Notice how there is a relationship between the angle and arc length in the first sector. If the radius doubled, so too would the arc length. If the angle doubled, the arc length would double as well! So arc length, $L = z \times r$ where z is the angle measured in radians.

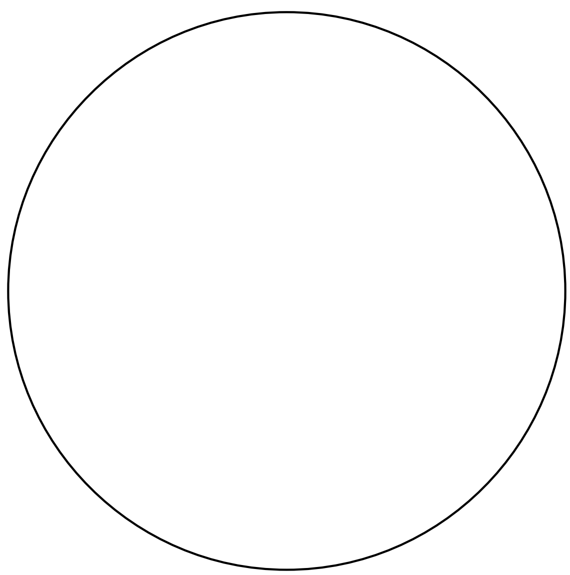
There is a similar equation for the area of a sector. In fact, you need to use arc length in order to calculate the area. $A = \frac{r \times L}{2}$. Remember, that when calculating area or arc length, it is very important that your *angle is in radians, NOT degrees*!

Circles & Triangles

Recall that the two most important shapes that exist are the circle and the triangle. Both are actually very closely related. When we were investigating the number of sides or edges a circle has, remember that we only looked at *regular polygons*. When we **inscribed** each polygon in a circle, all the corners or vertices were on the circumference of the circle.

Some irregular polygons can be inscribed so that this property (of vertices intersecting the circumference) holds. Simply select a number of points on the circumference of a circle, and draw chords between all adjacent points.

Draw an irregular hexagon which is inscribed in the circle below.



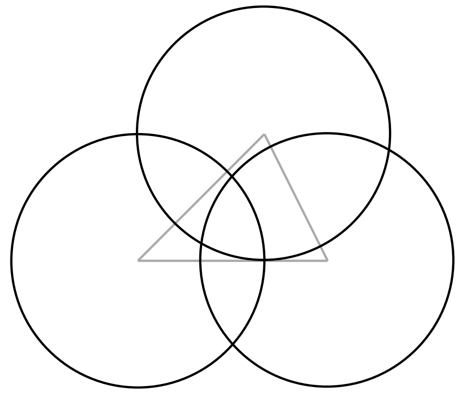
However, there are many irregular polygons which cannot be inscribed where all vertices intersect with the circumference. The one exception is the triangle. *Every* triangle can be inscribed by a circle so that all three vertices intersect with the circumference.

To inscribe a triangle in a circle, we will need two tools: a **compass** and a **ruler**, as well as a triangle to be inscribed. Use the following process to inscribe a triangle.

1. Observe the longest edge of your triangle. If two or all three edges are similar in length, simply select one.

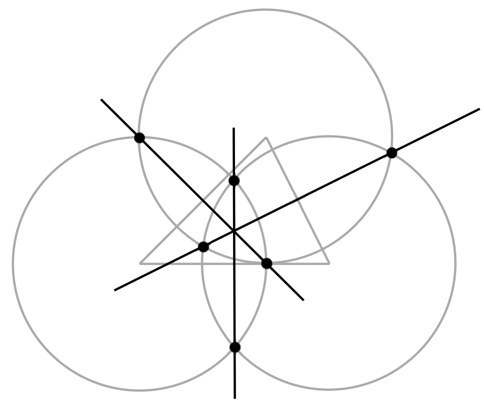
2. Separate the legs of the compass so that the distance between the point and pencil is about $\frac{3}{4}$ the length of the line you selected in *Step 1*.

3. Using your compass, draw 3 circles, each one centred at a vertex of the triangle. *Be sure not to alter the spread of your compass*! The 3 circles will cross over one another. You should have something like this:



4. Choose 2 of the circles you have drawn. The circumferences of these circles will intersect in two locations. *Using a ruler*, draw a line which connects both points. Ensure this line is extended past these points.

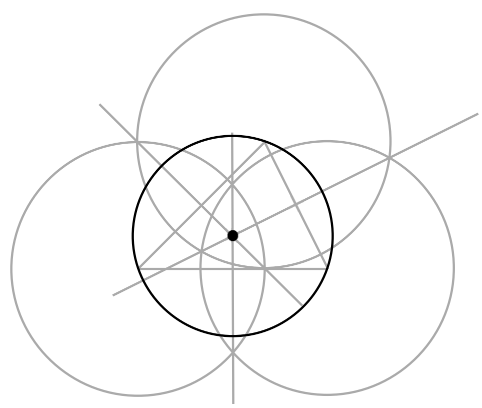
5. Repeat *Step 4* with the other two combinations of circles. You should get this:



6. Each line is actually *perpendicular* to one of the edges of the triangle. As well, it divides that same edge into 2 *equal* halves. This is why these lines are called **perpendicular bisectors**.

7. Notice that the 3 perpendicular bisectors intersect at one point. This will always happen if you draw your lines correctly! This intersection point is called the **circumcentre**. It becomes the centre of the circle which inscribes the triangle.

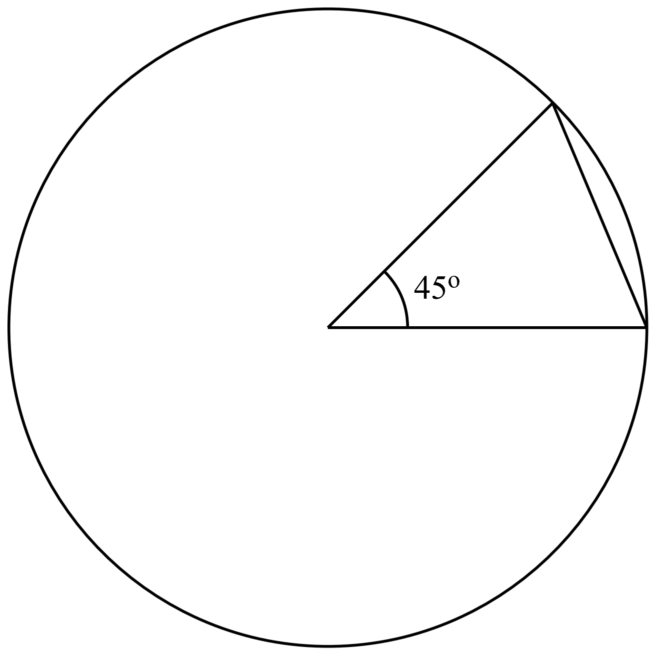
8. Place the point of your compass on the circumcentre. Extend the other leg to any of the vertices of the triangle. Now draw the circle around the circumcentre, and it will inscribe the triangle.



We have dealt with many theoretical applications of circles. There are SO MANY MORE that we simply don't have time to cover! By completing the problem set you will see some of the applications of the theory we did cover.

Problems

1. Given the following circle, measure the radius and chord length, then calculate the diameter, circumference, arc length, area of the whole circle, and the area of the sector.

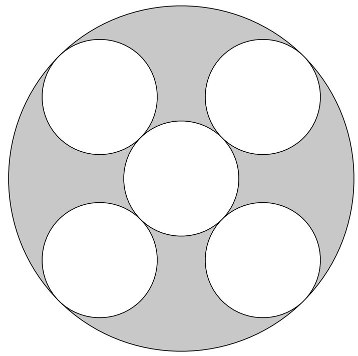


2. Given the following radii and angle between them, calculate the arc length and sector area.

   (a) 3cm, 2 rad (b) 2m, 135° (c) 25cm, 60°

3. Circle C, with centre O, has two radii which are 5cm long. One radii ends (on the circumference) at point A and the other ends at point B (also on the circumference). If angle AOB is $\frac{\pi}{2}$rad, what is the length of chord AB? (Hint: Draw a diagram and also use Pythagorean Theorem)

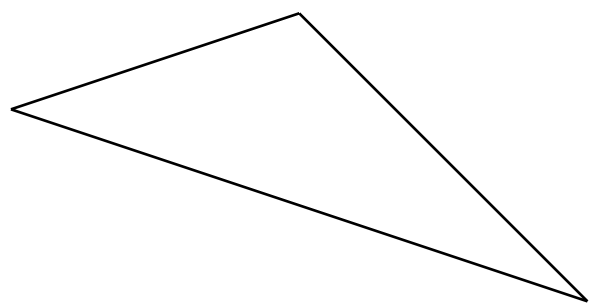
4. Given the following figure, what is the area of the shaded region if the diameter of the large circle is 6cm and all of the smaller circles have the same radius? What is the radius of a circle with an equivalent area?



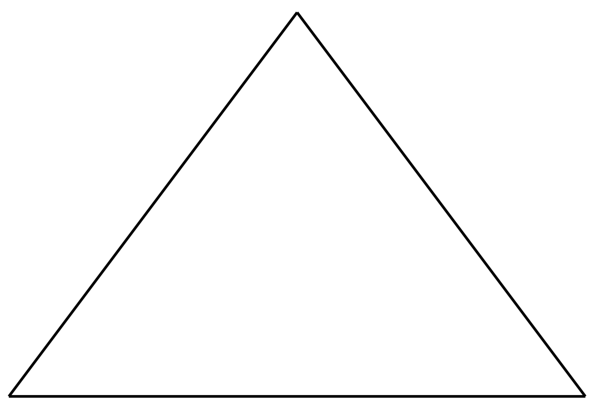
5. CHALLENGE: What is the total area of all the darker shaded petals, if all the circles have a radius of 2cm? Note all the petals are the same size and don't overlap one another. Also the vertices of each petal intersect with three other petal vertices.

6. Inscribe the following triangles in a circle.

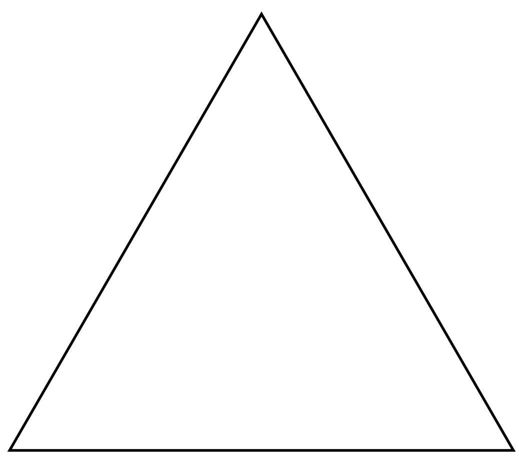
(a)



(b)



7. Take the following equilateral triangle and find the circumcentre C, as normal. Draw the circle which inscribes the triangle. Also draw another circle, centred at C, which is *inscribed by* the triangle. You will notice that the sides of the triangle are tangent to the circle at the points where the perpendicular bisectors intersect with the triangle's sides.



8. Given the irregular octagon below, find the circumcentre using the same process as for triangles. Then inscribe this figure in a circle.