UNIVERSITY OF
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Waterloo, Ontario N2L 3G1

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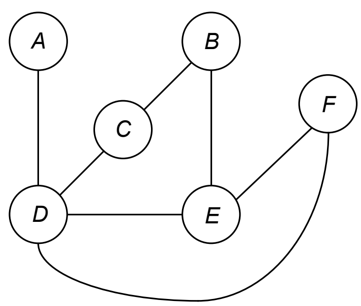
Grade 7 & 8 Math Circles
Graph Theory
March 5/6, 2013

Not that Kind of Graph

When you hear the word graph, most people will think of a bar graph, a line graph or something similar. There is another type of graph in mathematics. These graphs have points or **vertices** and lines between vertices called **edges**. An example of a graph is given below. The vertices are shown with circles and labelled with letters, and the edges are the lines between the vertices.



Vertices and Edges

Usually vertices have some sort of label, which can be people, places, numbers, colours, sports, almost *anything*! This is great for referring to a specific vertex. For example, in the graph above we can say vertex A and D are joined by an edge. Since we've used the labels A and D we know which vertices we're talking about. With that, notice how two vertices cannot be labelled with the same name.

Edges must go between two different vertices: They cannot 'loop' back to the vertex where it started and cannot end anywhere but a vertex. Edges can show that a relationship exists between the pair of the joined vertices. For example, an edge between two people might show that they are relatives. However in general graphs, like the one above, edges merely connect vertices and the relationship is unknown.

If two vertices have an edge between them they are said to be **adjacent**. All vertices that are adjacent to vertex E are called the **neighbours** of E. The **degree** of a vertex is the number of neighbours a vertex has.

In the graph above, what are the neighbours of vertex E? What is the degree of vertex E?

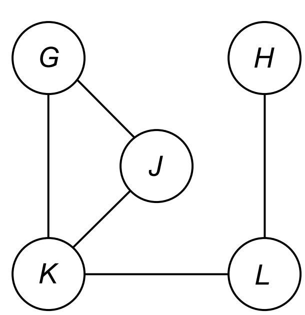
We know that we can refer to a specific vertex by its label. What about the edges of a graph? Do edges also have specific labels? Yes! We can refer to an edge by the two vertices it is joined to. However there is a specific notation we must use with **curly brackets**, '{' and '}'. For example, edge {A,D} is the edge which joins vertices A and D.

Write all edges which join a vertex to E from the above graph.

An important note about this edge notation is that the order of vertices *does not matter*. Edge {A,D} is the same as edge {D,A}. Also when we draw a graph edges can be curved, and can cross over one another. Finally, there can never be more than one edge between two vertices.

Paths and Connected Graphs

Let's look at another graph:



Notice how there isn't an edge connecting vertices H and J. But H is adjacent to L, L is adjacent to K, and K is adjacent to J. So in a way H and J are connected even though they aren't adjacent. The sequence of vertices and edges going from H to J is called a **path**.

When writing a path, start at a vertex, write the connecting edge on the path, followed by the adjacent vertex. Continue writing connecting edges and vertices until you reach the final vertex.

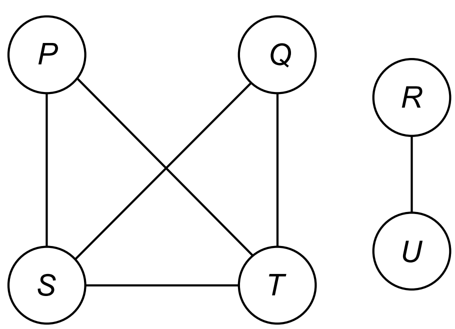
The path from H to J above, written formally is: H{H,L}L{L,K}K{K,J}J. There is also short way to write this. Simply list out the vertices as you pass over them. So the short form version of the path is HLKJ.

Now for a very important rule: a path cannot contain the same vertex more than once. So the following is **NOT** a path in the above graph: HLKLKJ

But Wait! Try to find a different path from H to J. ______________________________
A different path will have a different combination of adjacent vertices starting with H and ending with J, as well as not repeating any vertex.

The **length** of a path is the number of *edges* in the path. So the length of our path HLKJ is 3.

In some graphs there may not be a path between two vertices. For example:



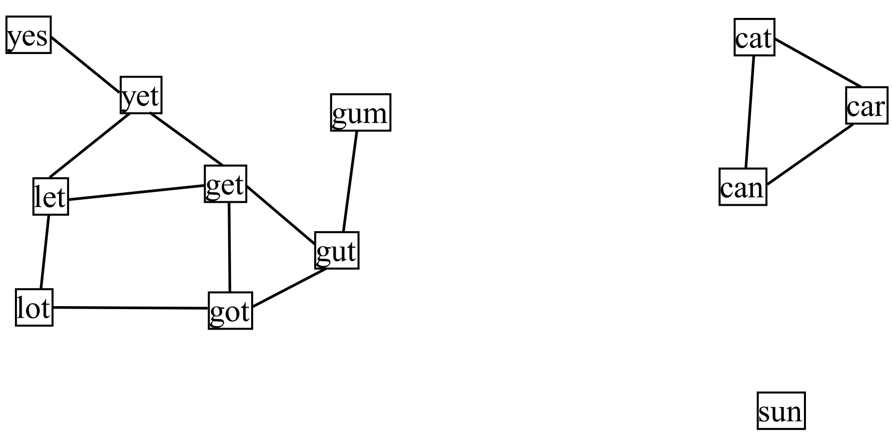
We say that a graph is **connected** if there is a path between every pair of vertices. In the above example, the graph in not connected since there is no path between the pair of vertices Q and R (also others).

But in this example, P, Q, S & T are all connected (since there are paths between all these numbers) and R & U are connected. All of the vertices in this graph are connected to others within a section of the graph. These connected sections of a graph are called the **components** of the graph. So the above graph has 2 components in it. A connected graph, like the first two examples, will only ever have one component.

A quick note: In this graph edges {P,T} and {Q,S} cross one another. However, remember that an edge can only go between two vertices. So in a drawing of a graph, an edge will continue in the same direction after an intersection. Thus, there is no edge between P and Q (although there is a path).

Now lets look at an example of a specific graph.

Example: A word graph has vertices labelled with words of the same length. Two words are adjacent if you can change exactly one letter in one word to get the other. Below is part of a word graph which we will call W.



1. How many components does W have? __________

2. What is the degree of the vertex "get"? ________ What about "sun"? ________

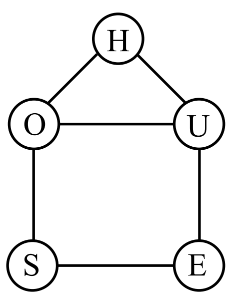
3. Give two different paths from “gut” to “yes”. What is the length of each path?

4. Add some more vertices and edges so that W becomes a connected graph.

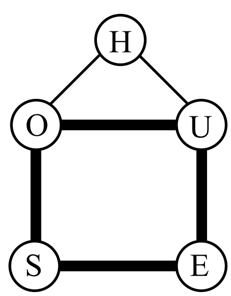
Cycles

Remember that a path cannot contain a vertex more than once. However there is a special type of path we will look at that breaks this rule.

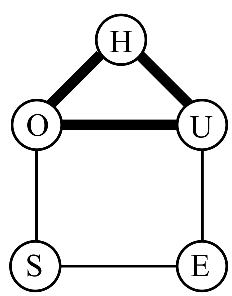
A **cycle** is a path which begins and ends on the same vertex, with no other vertices repeated. In the graph HOUSE below there are 3 different cycles. Note how in each cycle the first and last vertices are the same.



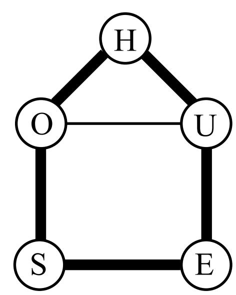
Graph HOUSE



Cycle OUESO



Cycle HOUH



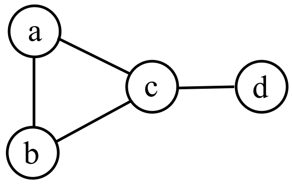
Cycle HUESOH

Just as with paths, the **length** of a cycle is the number of edges it contains.

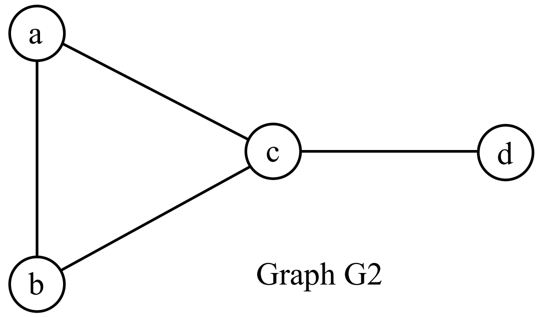
In the last cycle, notice how all vertices are included in the cycle. A **Hamilton cycle** is a cycle that includes all vertices of a graph, or alternately the length of the cycle equals the number of vertices in the graph. Many graphs will not have a Hamilton cycle.

Isomorphic Graphs

Let’s take a look at some graphs.



Graph G1

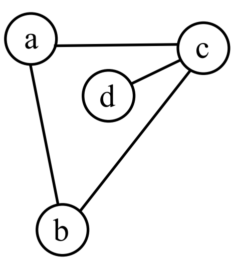


Graph G2

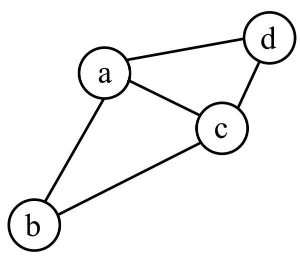
Are G1 and G2 the same graph? Yes! The vertices on both are labelled the same. Remember that edges just connect two vertices. So even though the lengths of the edges are longer in G2, *it has no meaning*!

This is actually a good quality of graphs. If I gave you a list of vertices and edges, and asked you to draw the graph, no two people would draw the graph with exactly the same lengths of edges.

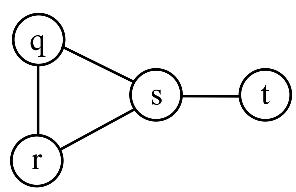
With that, are G3, G4, and G5 the same graphs as G1?



Graph G3



Graph G4



Graph G5

Hopefully you can see that G3 has all the same vertices and edges as G1. So the position of vertices in a drawing of a graph doesn't matter so long as the edges connect the proper vertices! Again if everyone drew a graph given only a list of vertices and edges, there would be many different ways it could be drawn, yet we want these to be the same graphs.

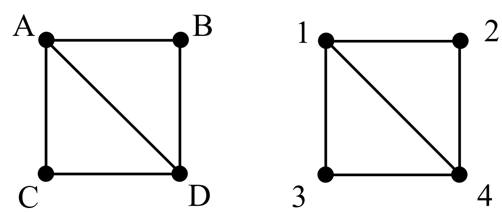
Observing G4, note there is an extra edge ({a,d}) not in G1. So G1 and G4 are not the same graph because they don't have the same edges.

In G5 the vertices are labelled differently. So the vertices are not the same, thus G1 and G5 are not the same graph. However, looking at G1 and G5, there are many things in common. In fact, other than the labels on the vertices, these graphs are the same: corresponding vertices have the same degree and are adjacent to the same corresponding vertices.

Two graphs G_1 and G_2 are **isomorphic** if vertices u & v are adjacent in G_1 and, *f(u)* & *f(v)* are adjacent in G_2. *f(u)* & *f(v)* just mean we've changed the vertex labels from u & v to new labels.

Graphs which are isomorphic to one another will also share many properties. The length and number of cycles will be the same in both cases, for example.

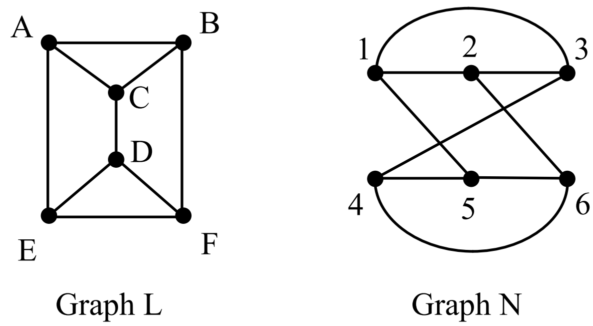
Now lets look at a few examples:



Graph Alpha Graph Num

As you can see, these two graphs are identical when we ignore the labels. So Alpha and Num are isomorphic graphs. We can even write out the change in the labels: *f(A)*=1 (this means we changed vertex A to vertex 1), *f(B)*=2, *f(C)*=3, and *f(D)*=4. If we wanted to, we could also show that the edges changed but are still connected to the corresponding vertices. One example is $f(\{A,B\})=\{1,2\}$.

But this was a fairly obvious isomorphic example. Are the following graphs isomorphic?



This is much harder. At first glance it may look as though L and N are not isomorphic since they don't look too similar. But remember that the same graph can be drawn in many different ways! In fact, these graphs are isomorphic. Prove this by defining the renaming of each vertex using $f()$, and show it works for all the edges.

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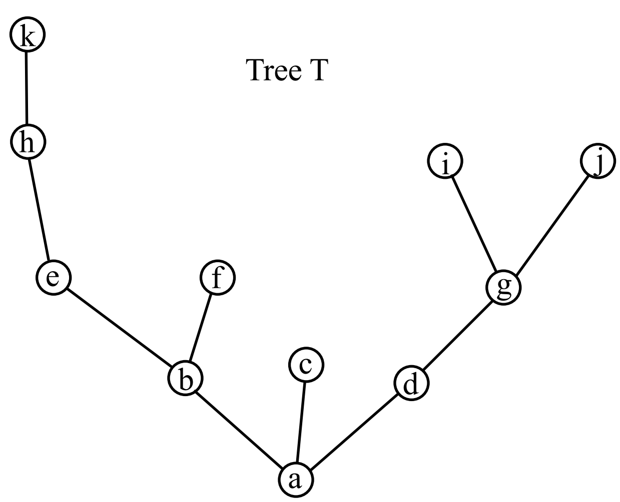
Trees

Some graphs won't have any cycles. A connected graph with no cycles is called a **tree**. Below is an example of a tree.



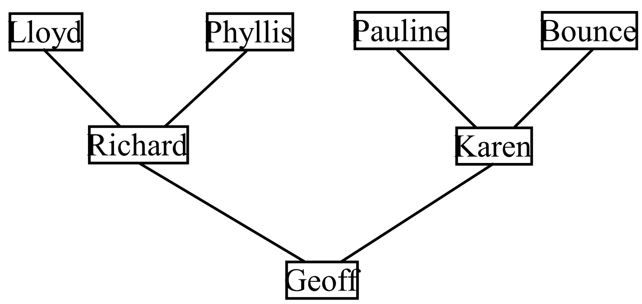
In the drawing of a tree, usually one vertex will be at the bottom (or top) and all other vertices will branch out from this vertex. This vertex is called the **root** of a tree.

Oddly enough, every vertex in a tree can be the root. On a separate piece of paper redraw tree T with vertex b as the root then again with vertex g as the root.

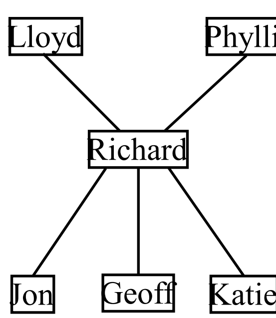
Remember that though these graphs look different, they are in fact the same graph.

When we look at a tree, some vertices are further from the root. The **level** of a vertex is the length of the path from the the root to the vertex. So in the first example, level(a) = 0 and level(h) = 3.

One real world example of a tree, is your family tree. Put yourself as the root with the next level as your parents, then your grandparents and so on. Edges between people show a parent/child relationship. Here's part of my family tree:



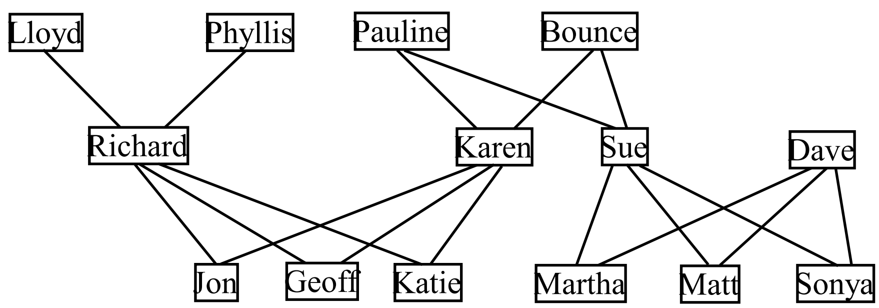
We can also start with a different root. Here's part of my father's family tree (note these are not the same tree!):



Notice how this tree shows the root in the middle. The names below the root show children, and names above show parents. When making a family tree it is important to have some form of order (parents of a vertex above and children of a vertex below) even though the placements of the vertices doesn't matter in a normal graph. Despite this difference both Jon and Phyllis are on level 1 of the tree.

It's also possible to combine multiple family trees into one. However, when this occurs the tree actually becomes a general graph since cycles occur, and there is no obvious root.

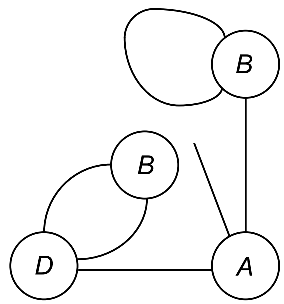
Here's part of my family graph:



Graphs can be used to represent so many different things: people, places, events, relationships and connections. In fact there are entire courses and fields of study devoted to graph theory! The applications of graphs are widespread and abundant. What are some other connections or relationships that could be represented by graphs?

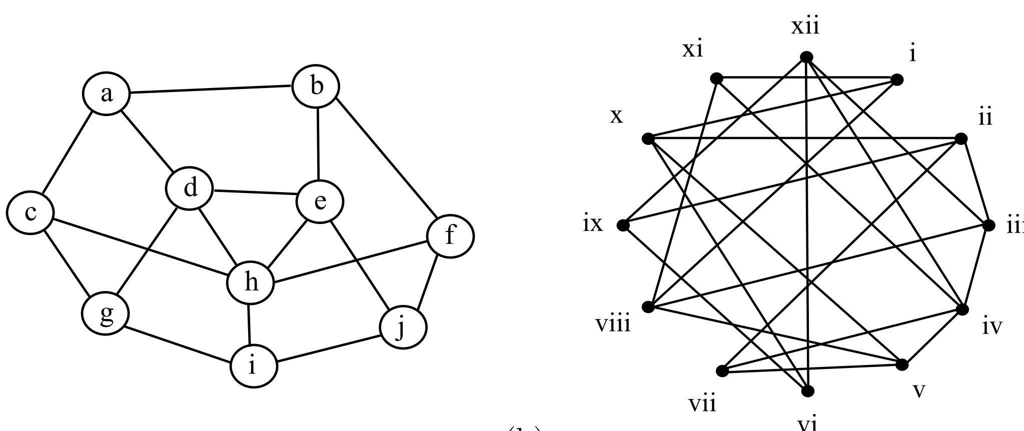
Problems

1. Explain all the reasons why the drawing below is not a graph.



2. Create a word graph like the one from the lesson. Ensure it is a connected graph and contains the following words: log, top, eat, mud and rag.

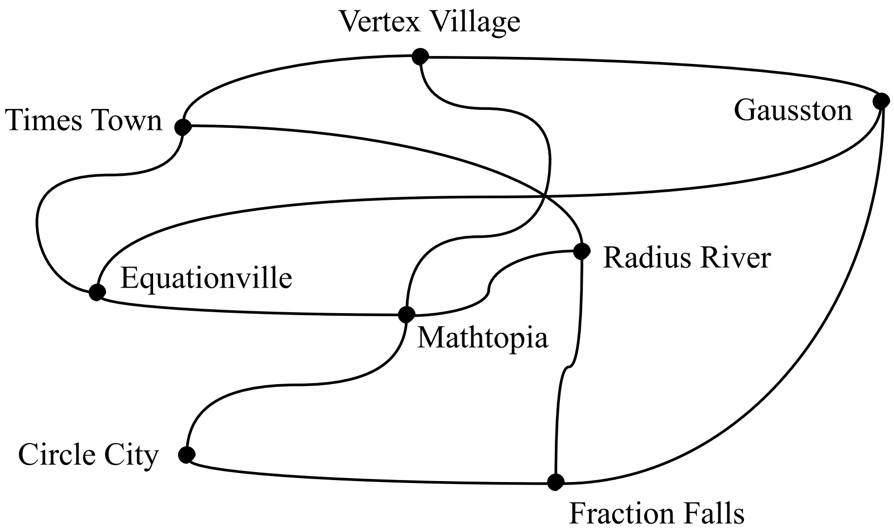
3. Find a Hamilton cycle in the following graphs.



(a)

(b)

4. Euler's Oil is a gas station chain in the country of Mathisfun. The CEO of the company, Leonhard, wants to visit all his gas stations in the country to ensure that each one is stocked with enough fuel. However, he doesn't want to visit any town more than once as he is a very busy man. Suggest a route Leonhard could take from the head office in the capital Mathtopia, through each town exactly once, and return to the capital. Note that despite the intersection of three highways in the centre of the country, there is no way to change from one road to another (i.e. there is no direct road from Mathtopia to Gausston, etc.)



5. In a digit graph, numbers with the same number of digits not beginning with 0, are vertices. Two numbers are adjacent if exactly one digit is different. So 136 and 836 are adjacent but 549, 550, and 459 are not.

   (a) Let D_2 be the digit graph of numbers from 10 to 99. At most, what is the shortest path between any two numbers?

   (b) Let D_5 be the digit graph of numbers with 5 digits. At most, what is the shortest path between any two numbers?

   (c) Suggest one path that is as short as possible from number 142 874 to 985 304 in D_6.

   (d) What is the degree of vertex 139 in D_3

6. Given is a map of South America. Create a graph with the vertices representing countries and the edges joining those countries which share a land border. So Chile and Peru are adjacent, but Peru and Uruguay are not. Use your graph to answer the questions below.

Courtesy of abcteach.com

(a) When trading goods over land, a \$100 tax is paid to each country which the goods travel through. So if Columbia sells coffee to Venezuela, the least amount of tax paid is \$100.

   i. What is the least amount of tax paid on wool shipped from Ecuador to Paraguay?

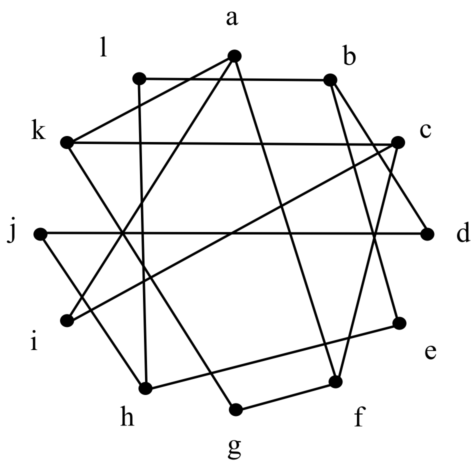
   ii. What country can trade with the most others for exactly \$100?

   iii. Brazil raises it's taxes to \$400. What is the cheapest amount of taxes paid to ship Venezuelan oil to Paraguay? What about French Guiana to Bolivia?

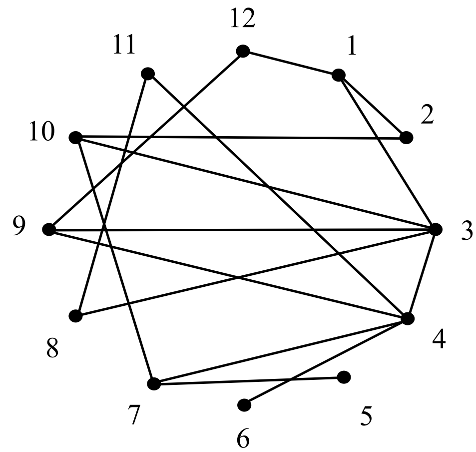
   iv. To encourage trade, Peru will not tax any goods going though the country. Chile and Suriname want to trade lumber and wheat. What route will result in the least amount of taxes? How much will they pay?

(b) Kevin wants to visit every country in South America on his vacation. He doesn't want to visit any country more than once since crossing the border can take a long time! He will fly into Lima, Peru to begin his trip and fly out of Montevideo, Uruguay. In what order should he visit each country?

7. CHALLENGE: Explain why there are no Hamilton cycles in the following graphs.

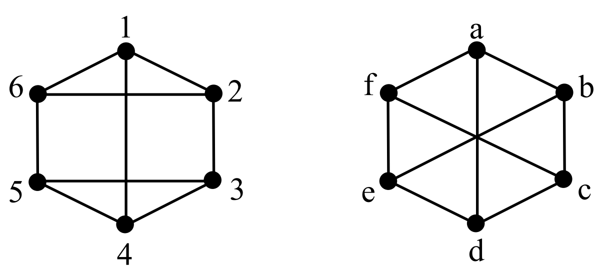


(a)

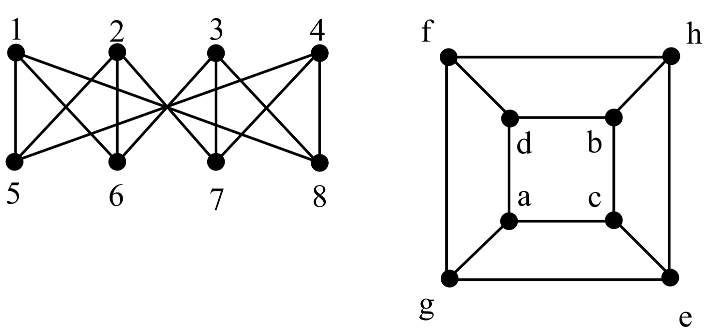


(b)

8. Determine if the following pairs of graphs are isomorphic. If so, define the renaming of each vertex using $f()$, or explain why not.



(a)



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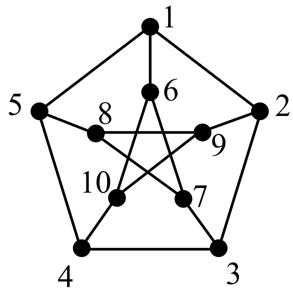
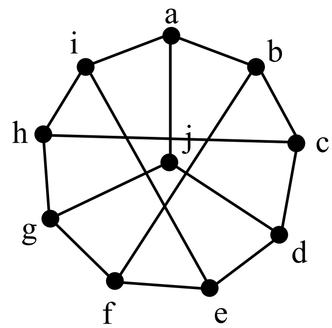




(c)

9. (a) Create a family tree with you as the root.

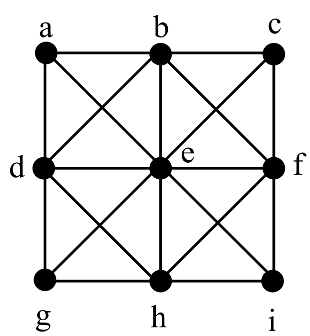
   (b) Now create a family web, showing siblings, cousins, aunts, uncles and other extended family members.

10. CHALLENGE: Solve each question and *explain why* your answer is correct.

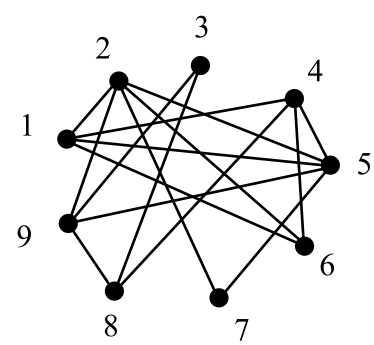
    (a) If V = number of vertices in a tree, how many edges are there?

    (b) If there are 20 vertices in a graph and each vertex has a degree of 3, how many edges are in the graph?

11. A **planar graph** is a graph which can be drawn so that no vertices cross. Redraw each graph to show it is planar.



(a)



(b)